Certificate Course on Complex Analysis

Organized
By
Department of Humanities and Sciences

Course Duration: 26/10/2021 to 02/12/2021

Course Coordinator: Dr.B.Rama Bhupal Reddy

Course Instructors: Dr.B.Rama Bhupal Reddy
Dr.G.Radha
Sri.B.Veera Sankar

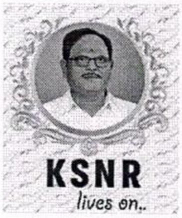
K.S.R.M. COLLEGE OF ENGINEERING

(UGC-AUTONOMOUS)

Kadapa,Andhra Pradesh, India– 516 003

Approved by AICTE, New Delhi & Affiliated to JNTUA, Ananthapuramu.

An ISO 14001:2004 & 9001: 2015 Certified Institution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Lr. /KSRMCE/ (Humanities & Sciences)/2021-22/

Date: **19.10.2021**

To

The principal,
K.S.R.M.College of Engineering
Kadapa.

From

Dr. B.Rama Bhupal Reddy,
Professor of Mathematics,
Department of H&S,
K.S.R.M College of Engineering,
Kadapa.

Respected Sir,

Sub: KSRMCE - Department of H&S (Mathematics) Permission to conduct Certificate course on Complex Analysis– Request –Reg.

With reference to the above subject, it is brought to your kind notice that, the H&S Department is planning to conduct a Certificate Course on **Complex Analysis** for B. Tech III-Sem students from **26th October 2021 and ends on 2nd December 2021**in Offline mode. In this regard I kindly request you sir to grant permission to conduct certificate course. This is submitted for your kind perusal.

Thanking you Sir,

Yours Faithfully
B. Ram
Dr.B.Rama Bhupal Reddy,
Professor of Mathematics,
Dept. of H&S,
K.S.R.M.C.E.

Forwarded to
Principal

Permitted
V.S.S. Murthy

K.S.R.M. COLLEGE OF ENGINEERING
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Cr./KSRMCE/(Department of H&S)/2021-22 Date:19-10-2021

Circular

It is here by informed that the Department of H&S is going to conduct certificate course on Complex Analysis to B. Tech III-Sem students. This certificate course starts from 26th October 2021and ends on 2nd December 2021. Interested students may register their names with the following link before 25thOctober 2021.

Registration Link: https://forms.gle/4TQhm4MLnmJVN5iz8

For any queries contact,

Convener
Dr.I.Sreevani, HoD, H&S

Coordinators

Dr.B.RamaBhupal Reddy, Professor, Dept. of H&S, (Ph. No: 9490032642)
Dr.G.Radha, Assoc.Prof, Dept. of H&S, (Cell No:9966815484)
Sri.B.VeeraSankar, Asst. Prof, Dept. of H&S, (Cell No: 9966072081)

HOD H&S

Dr. I. SREEVANI M.Sc.,Ph.D.
Head of Humanities & Sciences
K.S.R.M College of Engineering
KADAPA - 516 005

Cc to:

The Management /Deans/HoDs/IQAC for information

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Date: 20-10-2021

Name of the Event: Certification Course on Complex Analysis

Venue : CE-109

Registration Form

S. No	Roll Number	Name of the student	Department	Signature
01	19941A0139	P. Manjunath	civil	P. Manjunath
02	199Y1A0140	R. vineeth Reddy	civil	R. vineeth
03	199Y1A0158	S. kavitha	civil.	S. Kavitha.
04	199Y1A0161	G. Surendra	civil	G. Surendra
05	199Y1A0164	T. Anil kumar Redd	Civil	T. A
06	199Y1A0148	S. surendra	Civil	S. surendra.
07	209Y5A0131	K. Dharmateja.	civil	
08	199Y1A0150	S. Aswak	civil	S. Aswak.
09	199Y1A0151	S. Baba Vazeeru	Civil	S. Vazeer
10	199Y1A0153	S. Imran	Civil	Imran Shaik
11	199Y1A0147	S. Sudharshan	civil	S. sudharshan
12	199Y1A0172	Y. Bramhini	Civil	Y. Bramhini
13	209Y5A0116	D. Sivanithyan	Civil	D. S
14	19941A0156	S. M. D AAtif	civil -A	aatif
15	19941A0145	S. Pavan kuma reddy	Civil - A	S. Pavan.

16	209Y5A0145	M.V.Jagadeeshwar Reddy	Civil	Jagadeesh.
17	199Y1A0307	D.BharathSimhareddy	ME	D. Bharath Simha reddy
18	199Y1A0316	K.Naveen kumar Reddy	ME	K.Naveen kumar Reddy
19	209Y5A0114	C.Prudhviswarreddy	Civil	C.Prudhvi
20	209Y5A0103	B.Pavithra	Civil	
21	209Y5A0106	B.Sree Hari	Civil	B.Sree Hari
22	199Y1A0160	S.RajeshReddy	civil	S.Rajesh
23	199Y1A0311	G.Bharath	MEC	G.Bharath
24	199Y1A0334	P.Vamsidhar Reddy	M.E	P.Vamsidhar Reddy.
25	199Y1A0120	K. Kejiya	Civil	K. Kejiya
26	199Y1A0158	D.Anusha.	Civil	D.Anusha.
27	199Y1A0109	D.chennakeshava	civil	D.chennakeshava.
28	199YA0101	B.AviNash	civil	B.Avinash.
29	199Y1A0132	N.Seshasai	civil	N.Seshasai
30	199Y1A0126	M.Jaganmohan	civil	M.Jaganmohan
31	199Y1A0127	M.Yagna Priya	Civil	YP
32	199Y1A0106	C.Haritha	civil	C.Haritha
33	199Y1A0142	R.Yeshwanth Reddy	CIVIL	R.Ye
34	199Y1A0146	S.MOhAMMAD	CIVIL	sMaj
35	199Y1A0136	PArthaf khan	CIVIL	P.A
36	209Y5A0101	A.Supraja	Civil	A.Supr
37	209Y5A0126	J.V.S.prasanna	C.E	J.Sai prasanna.
38	199Y1A0352	S.Abhilash kumar Reddy	M.E	S.Abhilash
39	199Y1A0360	Y.Harshavardhan Reddy	M.E	Y.Reddy
40	199Y1A0301	A.Sreedhar	M.E	A.Sreedhar
41	199Y1A0329	M.Sreekanth Reddy	M.E	Sreekanth

42	199Y1A0340	S.D.Varma	ME	SD.varma
43	199Y1A0345	S.Md.mansoor	ME	Smm
44	199Y1A0359	Y.Saikumar Reddy	ME	Y.Saikumar Reddy
45.	199Y1A0303	B.SHIVA REDDY	ME	Baa siva Reddy

Co-ordinator

HOD H&S

Dr. I. SREEVANI M.Sc.,Ph.D.
Head of Humanities & Sciences
K.S.R.M. College of Engineering
KADAPA - 516 005

Course Title	**COMPLEX ANALYSIS (R20)**	**Certificate Course** CE & ME Branches

Course Objectives:

The concepts of complex variables to equip the students to solve application problems.

Course Outcomes: On successful completion of this course, the students will be able to	
CO 1	**Define** analytic function.
CO 2	**Analyze** images from z-plane to w-plane.
CO 3	**Determine** complex integration along the path.
CO 4	**Define** singularities, poles and residues.
CO 5	**Analyze** real definite integrals by residue theorem.

Module I:

Functions of a complex variable – Limit – Continuity -Differentiability – Analytic function – Properties – Cauchy – Riemann equations in Cartesian and polar coordinates – Harmonic and Conjugate harmonic functions. Construction of analytic function using Milne's - Thomson method.

Module II:

Conformal Mapping: Some standard transforms – translation, rotation, magnification, inversion and reflection. Bilinear transformation – invariant points. Special conformal transformations: $w = e^z$, z^2, sinz and cosz.

Module III:

Complex integration: Line integral - Evaluation along a path – Cauchy's theorem – Cauchy's integral formula - Generalized integral formula.

Module IV:

Singular point – Isolated singular point – Simple pole, Pole of order *m* – Essential singularity. Residues: Evaluation of residues. Cauchy's residue theorem.

Module V:

Evaluation of the real definite integrals of the type (i) Integration around the unit circle $\int_0^{2\pi} f(cos\theta, sin\theta) d\theta$ and (ii) integration around a small semi circle $\int_{-\infty}^{\infty} f(x)dx$

Text books:

1. Higher Engineering Mathematics, Dr. B.S Grewal, Khanna Publishers-42 edition.
2. Advanced Engineering Mathematics, Erwin Kreyszig, Willey Publications, 9th edition-2013.

1. Higher Engineering Mathematics, B.V.Ramana, Mc.Graw Hill Education (India) Private Limited.
2. Advanced Engineering Mathematics by N. Bali, M Goyal, Firewall Media 7th edition.
3. Engineering Mathematics, Volume – III, E. Rukmangadachari & E. Keshava Reddy, Pearson Publisher.

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K.S.R.M COLLEGE OF ENGINEERING, KADAPA

(Autonomous)

Yerramasupalli, Kadapa, Andhra Pradesh – 516003

Department of Humanities & Sciences

Certification Course
ON
Complex Analysis

Schedule

Date	Timing	Course Instructor	Topic to be covered
26-10-2021	4.00pm-5.00pm	Dr. B. Rama Bhupal Reddy	Introduction to Complex Numbers
27-10-2021	4.00pm-5.00pm	Dr.B. Rama Bhupal Reddy	Functions of a complex variable, Limit, Continuity, Differentiability
28-10-2021	4.00pm-5.00pm	Dr.B.Rama Bhupal Reddy	Analytic function and Properties.
29-10-2021	4.00pm-5.00pm	Dr.B.Rama Bhupal Reddy	Cauchy – Riemann equations in cartesian coordinates
30-10-2021	4.00pm-5.00pm	Dr.B.Rama Bhupal Reddy	Cauchy – Riemann equations in polar coordinates
01-11-2021	4.00pm-5.00pm	Dr.B.Rama Bhual Reddy	Problem solving based on Cauchy – Riemann equations
02-11-2021	4.00pm-5.00pm	Dr.B.Rama Bhupal Reddy	Harmonic and Conjugate harmonic functions
05-11-2021	4.00pm-5.00pm	Dr.B.Rama Bhupal Reddy	Construction of analytic function using Milne's Thomson method
06-11-2021	4.00pm-5.00pm	Dr.B.Rama Bhupal Reddy	Problem solving based on analytic function
08-11-2021	4.00pm-5.00pm	Dr.B.Rama Bhupal Reddy	Some standard transforms – translation, rotation, magnification, inversion and reflection
09-11-2021	4.00pm-5.00pm	Dr.B.Rama Bhupal Reddy	Bilinear transformation and invariant points
10-11-2021	4.00pm-5.00pm	Dr.B. Rama Bhupal Reddy	Problem solving based on Bilinear transformation and invariant points
11-11-2021	4.00pm-5.00pm	Dr.B.Rama Bhupal Reddy	Transformations: $w = e^z$, z^2
12-11-2021	4.00pm-5.00pm	Dr. B. Rama Bhupal Reddy	Transformations: w = sinz and cosz.
15-11-2021	4.00pm-5.00pm	Dr.G. Radha	Complex integration: Line integral - Evaluation along a path
16-11-2021	4.00pm-5.00pm	Dr.G. Radha	Problem solving based on Line integral along a path
17-11-2021	4.00pm-5.00pm	Dr.G. Radha	Cauchy's theorem
18-11-2021	4.00pm-5.00pm	Dr.G. Radha	Problem solving based on Cauchy's theorem
19-11-2021	4.00pm-5.00pm	Dr.G. Radha	Cauchy's integral formula

20-11-2021	4.00pm-5.00pm	Dr.G. Radha	Generalized integral formula.
22-11-2021	4.00pm-5.00pm	Dr.G. Radha	Problem solving based on Cauchy's integral formula
23-11-2021	4.00pm-5.00pm	Dr.G. Radha	Singular point, Isolated singular point, Simple pole
24-11-2021	4.00pm-5.00pm	Dr.G. Radha	Pole of order *m* – Essential singularity
25-11-2021	4.00pm-5.00pm	Sri. B.Veera Sankar	Evaluation of residues by formula
26-11-2021	4.00pm-5.00pm	Sri. B.Veera Sankar	Problem solving evaluation of residues by formula
27-11-2021	4.00pm-5.00pm	Sri. B.Veera Sankar	Cauchy's residue theorem
29-11-2021	4.00pm-5.00pm	Sri. B.Veera Sankar	Problem Solving based on Cauchy's residue theorem
30-11-2021	4.00pm-5.00pm	Sri. B.Veera Sankar	Introduction to evaluation of the real definite integrals of the type
01-12-2021	4.00pm-5.00pm	Sri. B.Veera Sankar	Evaluation of the real definite integrals of the type Integration around the unit circle
02-12-2021	4.00pm-5.00pm	Sri. B.Veera Sankar	Evaluation of the real definite integrals of the type integration around a small semi circle

Coordinator

HOD H&S

Dr. I. SREEVANI M.Sc.,Ph.D.
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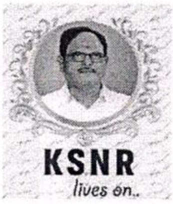
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lives on..

Date: 26-10-2021

Name of the Event: Certification Course on Complex Analysis

Venue : CE-109

List of Participants

S. No	Roll Number	Name of the student	Department	Signature
1	199Y1A0101	BOGGITI AVINASH KUMAR	CE	B. Avinash
2	199Y1A0106	CHINAMADULA HARITHA	CE	C. Haritha
3	199Y1A0109	DIRASANTHA CHENNAKESHAVA	CE	
4	199Y1A0119	KATUBOINA VEKRISHNA YADAV	CE	K yadav.
5	199Y1A0122	KUMBHAGIRI NAGARATHNA	CE	K. Nagarathna
6	199Y1A0126	MIDDE JAGAN MOHAN	CE	
7	199Y1A0127	MORAM YAGNA PRIYA	CE	YP
8	199Y1A0132	NAGA SESHA SAI	CE	N. Seshasai
9	199Y1A0136	PHATAN ARFATHULLA KHAN	CE	P. Arfathullakhan.
10	199Y1A0139	POOLA MANJUNATH	CE	P. Manjunath
11	199Y1A0142	RAMIREDDY YASWANTH REDDY	CE	yaswanthreddy
12	199Y1A0146	SALIVEMULA MAHAMMAD	CE	mohammad.
13	199Y1A0158	SIRANGI KAVITHA	CE	S. Kavitha.
14	199Y1A0161	SURABOINA SURENDRA	CE	S. Surendra
15	199Y1A0172	YEDDULA BRAMHINI	CE	Y. Bramhini

16	209Y5A0101	SUPRAJA	CE	
17	209Y5A0113	NAGA VENKATA THARUN KUMAR	CE	Tharun
18	209Y5A0126	VENKATA SAI PRASANNA	CE	
19	209Y5A0145	VENKATA JAGADEESHWAR REDDY	CE	
20	199Y1A0301	AKULA SREEDHAR	ME	A. Sreedhar
21	199Y1A0303	BANDI SHIVA REDDY	ME	
22	199Y1A0307	DEVAPATLA BHARATH SIMHA REDDY	ME	Bharath simha
23	199Y1A0311	GANUGAPENTA BHARATH	ME	G. Bharath
24	199Y1A0316	KETHIREDDY NAVEEN KUMAR REDDY	ME	
25	199Y1A0329	MOLAKALA SREEKANTH REDDY	ME	
26	199Y1A0334	PALLETI VAMSIDHAR REDDY	ME	Vamshi
27	199Y1A0340	SAGIRAJU DILLI VARMA	ME	
28	199Y1A0345	SHAIK MAHAMMED MANSOOR	ME	Shaik
29	199Y1A0352	SUDA ABHILASH KUMAR REDDY	ME	Harsha vardhan
30	199Y1A0359	YANDAPALLI SAI KUMAR REDDY	ME	Y. Saiku
31	199Y1A0360	YARRAPUREDDY HARSHAVARDHAN REDDY	ME	Harshavardhan

Co-Ordinator

HOD H&S

Dr. I. SREEVANI M.Sc.,Ph.D.
Head of Humanities & Sciences
K.S.R.M. College of Engineering
KADAPA-516 005

K.S.R.M College of Engineering (Autonomous), Kadapa

Department of Humanities & Sciences

Certification Course on Complex Analysis

Attendance Sheet

S. No	Roll No	Name of the student	26 10 21	27 10 21	28 10 21	29 10 21	30 10 21	1 11 21	2 11 21	5 11 21	6 11 21	8 11 21	9 11 21	10 11 21	11 11 21	12 11 21	15 11 21	16 11 21	17 11 21	18 11 21	19 11 21	20 11 21	22 11 21	23 11 21	24 11 21	25 11 21	26 11 21	27 11 21	29 11 21	30 11 21	1 12 21	2 12 21
1	19941A0101	Boggili A Vinash Kuma				A			A					A						A				A						A		
2	19941A0106	C. Haritha	Haritha	Haritha	Haritha	Haritha	A	Haritha	Haritha	Haritha	Haritha	A	Haritha	Haritha	Haritha	Haritha	Haritha	Haritha	A	Haritha	Haritha	Haritha	Haritha	Haritha	Haritha	Haritha	A	A	Haritha	Haritha	Haritha	Haritha
3	19941A0109	D. Chennakeshav	DCK	DCK	DCK	A	DCK	DCK	DCK	A	DCK	DCK	DCK	A	DCK	DCK	DCK	A	DCK	A	DCK	DCK	DCK	A	DCK	DCK	DCK	DCK	A	DCK	A	DCK
4	19941A0119	K. Krishna Yadav	K.Krishna	A	K.Krishna	K.Krishna	K.Krishna	K.Krishna	K.Krishna	A	K.Krishna	K.Krishna	K.Krishna	A	K.Krishna	K.Krishna	K.Krishna	A	K.Krishna	K.Krishna	A	K.Krishna	K.Krishna	K.Krishna	K.Krishna	K.Krishna	A	K.Krishna	K.Krishna	K.Krishna	A	K.Krishna
5	19941A0122	K. Nagarathna	Naga ratna	A	Naga ratna	Naga ratna	Naga ratna	Naga ratna	Naga ratna	A	Naga ratna		Naga ratna	A	Naga ratna	Naga ratna	Naga ratna	A	Naga ratna	Naga ratna	A	Naga ratna	Naga ratna	Naga ratna	A	Naga ratna	A	Naga ratna	Naga ratna	A	A	Naga ratna
6	19941A0126	M. Jagan Mohan	M. Jagan	M. Jagan	M. Jagan	A	M. Jagan	M. Jagan	M. Jagan	M. Jagan	A	M. Jagan	M. Jagan	M. Jagan	A	M. Jagan	A	M. Jagan	A	M. Jagan	M. Jagan	A	M. Jagan	M. Jagan	M. Jagan	A	M. Jagan	M. Jagan	A	M. Jagan	A	M. Jagan
7	19941A0127	M. Yagna Priya	M. Yagna priya	Yagna priya	Yagna priya	A	Yagna priya	Yagna priya	Yagna priya	Yagna priya	A	A	Yagna priya	Yagna priya	A	Yagna priya	A	Yagna priya	A	Yagna priya	Yagna priya	A	Yagna priya	Yagna priya	Yagna priya	A	Yagna priya	Yagna priya	A	Yagna priya	A	Yagna priya
8	19941A0132	N. Sesha Sai	N. Sesha Sai	N. Sesha Sai	A	N. Sesha Sai	N. Sesha Sai	N. Sesha Sai	A	N. Sesha Sai	N. Sesha Sai	N. Sesha Sai		N. Sesha Sai	N. Sesha Sai	N. Sesha Sai	A	N. Sesha Sai	N. Sesha Sai	N. Sesha Sai	A	N. Sesha Sai	N. Sesha Sai	N. Sesha Sai		N. Sesha Sai	N. Sesha Sai	N. Sesha Sai		N. Sesha Sai	A	N. Sesha Sai
9	19941A0136	P. Arfathulla Kha			A						A	A				A											A				A	
10	19941A0139	P. Manjunath	Manjunath	Manjunath	Manjunath	A	Manjunath	Manjunath	Manjunath	Manjunath	A	Manjunath	A	Manjunath	A	Manjunath	Manjunath	P.Manjunath	P.Manjunath	Manjunath	A	P.Manjunath	Manjunath	Manjunath	Manjunath	A	Manjunath	P.Manjunath	Manjunath	Manjunath	A	P.Manjunath
11	19941A0142	R. Yaswanth Reddy									A		A		A						A					A					A	
12	19941A0146	S. Mahammad				A				A	A		A		A						A					A				A	A	
13	19941A0158	S. Kavitha	Kavitha	Kavitha	Kavitha	Kavitha	A	Kavitha	Kavitha	Kavitha	A	Kavitha	Kavitha	Kavitha	Kavitha	A	Kavitha	Kavitha	Kavitha	Kavitha	A	Kavitha	Kavitha	A	Kavitha	Kavitha	Kavitha		Kavitha	Kavitha	Kavitha	Kavitha
14	19941A016	S. Surendra										A		A					A				A	A					A		A	
15	19941A0172	Y. Bramhini		A			A					A		A		A	A		A		A	A			A			A		A		
16	20945A0101	Supraja	Supraja	Supraja	A	Supraja	Supraja	Supraja	Supraja	A	Supraja	Supraja	Supraja	A	Supraja	Supraja	Supraja	Supraja	Supraja	A	Supraja	Supraja	A	Supraja	Supraja	Supraja	A	Supraja	Supraja	A	Supraja	Supraja
17	20945A0113	N.V. Tharun Kuma			A					A				A						A			A				A			A		
18	20945A0126	V. Sai Prasanna			A					A		A		A		A			A	A			A				A			A		
19	20945A0145	V. Jagadeeshwar Redd				A					A					A				A				A				A				
20	19945A0301	A. Sreedhar	Sreedhar	Sreedhar	Sreedhar	Sreedhar	A	Sreedhar	Sreedhar	Sreedhar	A	Sreedhar	Sreedhar	Sreedhar	Sreedhar	A	Sreedhar	Sreedhar	Sreedhar	A	Sreedhar	Sreedhar	Sreedhar	A	Sreedhar	Sreedhar	Sreedhar	A	Sreedhar	Sreedhar	Sreedhar	Sreedhar
21	20945A0303	B. Shiva Reddy	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy	A	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy	A	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy	A	B.Shiva reddy	B.Shiva reddy	A	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy	A	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy	A	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy	B.Shiva reddy
22	20945A0304	D. Bharath Simha Reddy									A					A			A					A				A				
23	20945A0311	G. Bharath	G.Bharath	G.Bharath	G.Bharath	G.Bharath	A	G.Bharath	G.Bharath	G.Bharath	A	G.Bharath	G.Bharath	G.Bharath	G.Bharath	A	G.Bharath	G.Bharath	A	G.Bharath	G.Bharath	G.Bharath	G.Bharath	A	G.Bharath	G.Bharath	G.Bharath	A	G.Bharath	G.Bharath	G.Bharath	G.Bharath

24	199Y1A0316	K.Naveen Kumar Reddy	K.Naveen	K.Naveen	K.Naveen	K.Naveen	A	K.Naveen	K.Naveen	K.Naveen	K.Naveen	A	K.Naveen	K.Naveen	K.Naveen	A	K.Naveen	K.Naveen	A	K.Naveen	A	K.Naveen	K.Naveen	K.Naveen	A	K.Naveen	K.Naveen	K.Naveen	A	K.Naveen	K.Naveen	K.Naveen
25	199Y1A0329	M.Sreekanth Reddy					A					A				A			A		A				A				A			
26	199Y1A0334	P.Vamsidhar Reddy					A					A				A			A		A				A				A			
27	199Y1A0340	S.Dilli Varma					A					A				A			A		A				A				A			
28	199Y1A0345	S.Mahammed Mansoor		A		A			A				A				A		A			A		A	A			A	A			
29	199Y1A0352	S.Abhilash Kumar Reddy		A		A			A				A						A			A			A			A	A			
30	199Y1A0359	Y.Sai Kumar Reddy		A		A			A				A						A			A			A					A	A	
31	199Y1A0360	Y.Harshavadhan Reddy				A		A	A			A	A		A		A	A		A		A		A		A		A		A		

G.Radha
COORDINATOR

HOD, H&S

Dr. I. SREEVANI M.Sc.,Ph.D
Head of Humanities & Sciences
K.S.R.M College of Engineering
KADAPA - 516 005

K.S.R.M. COLLEGE OF ENGINEERING
(UGC - Autonomous)
Kadapa, Andhra Pradesh, India - 516 003
Approved by AICTE, New Delhi & Affiliated to JNTUA, Ananthapuramu.
Department of H&S

Certification course on

"Complex Analysis"

Date
Starts from 26-10-2021

Eligibility: **CE & ME (V SEM)**

Venue: **CE 109**

Course Co-ordinator :
Dr.B.Rama Bhupal Reddy

Course Instructors:
Dr.B.Rama Bhupal Reddy
Dr.G.Radha
Sri.B.Veera Sankar

Dr. I.Sreevani, (HOD & Convener)
Dr. V.S.S. Murthy (Principal)
Prof. A. Mohan (Director)
Dr. Kandula Chandra obul Reddy (Managing Director)
Smt. K. Rajeswari (Correspondent, Secretary, Treasurer)
Sri K. Madan Mohan Reddy (Vice-chairman)
Sri K. Raja Mohan Reddy (Chairman)

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ACTIVITY REPORT

Certification Course

On

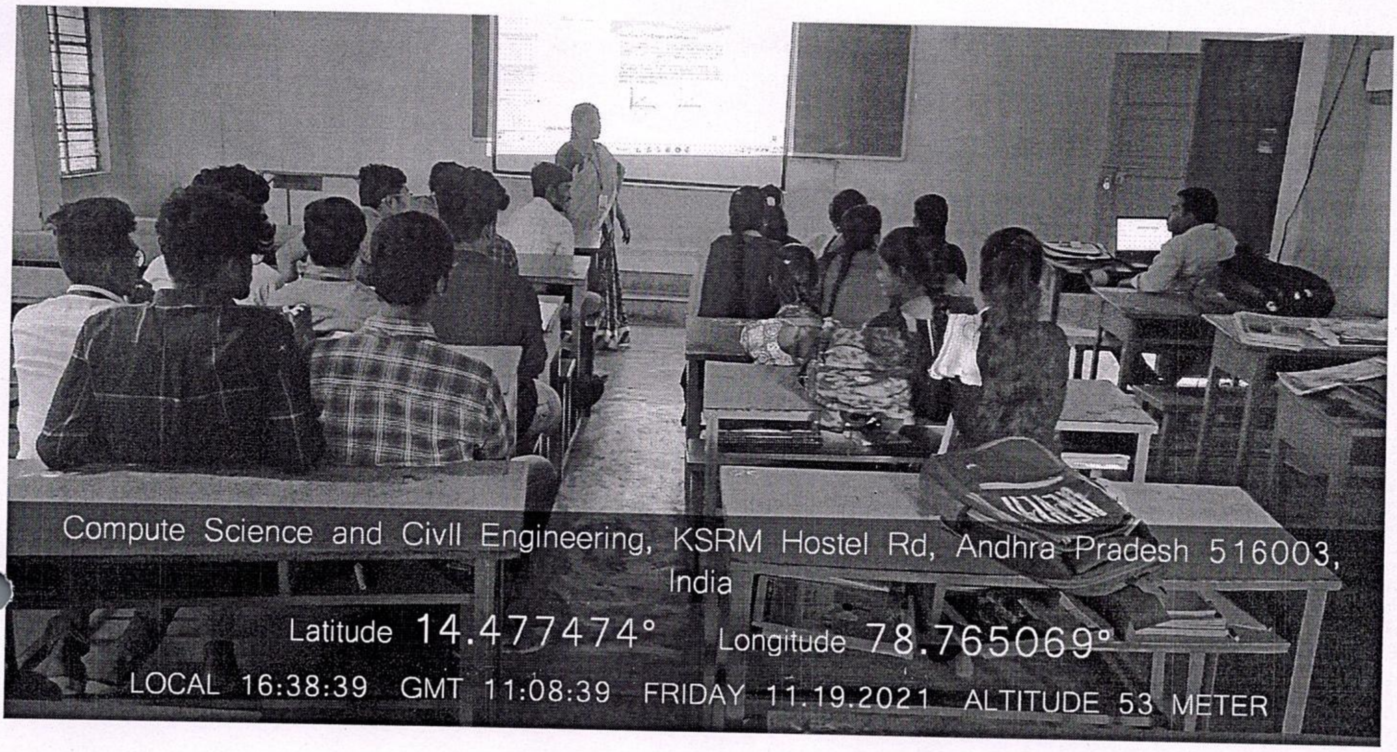
"*Complex Analysis*"

26th October 2021 **to** 2nd December 2021

Target Group	:	**Students**
Details of Participants	:	**31 Students**
Co-ordinator	:	Dr.B.RamaBhupal Reddy, **Prof, Dept. of H&S**
Organizing Department	:	**Department of Humanities & Sciences**
Venue	:	**CE-109**

Description: Certification course on Complex Analysis wasorganized by Department of Humanities and Sciences from26th October, 2021 to 2nd December, 2021 in offline mode. Dr.B.RamaBhupal Reddy, Dr.G.Radha and Sri.B.VeeraSankaracted as Course instructors. The main aim of the course is the study of functions that live in the complex plane,that is, functions that have complex arguments and complex outputs. Course was successfully completed and participation certificates were provided to the participants.

Photos



Compute Science and Civil Engineering, KSRM Hostel Rd, Andhra Pradesh 516003, India

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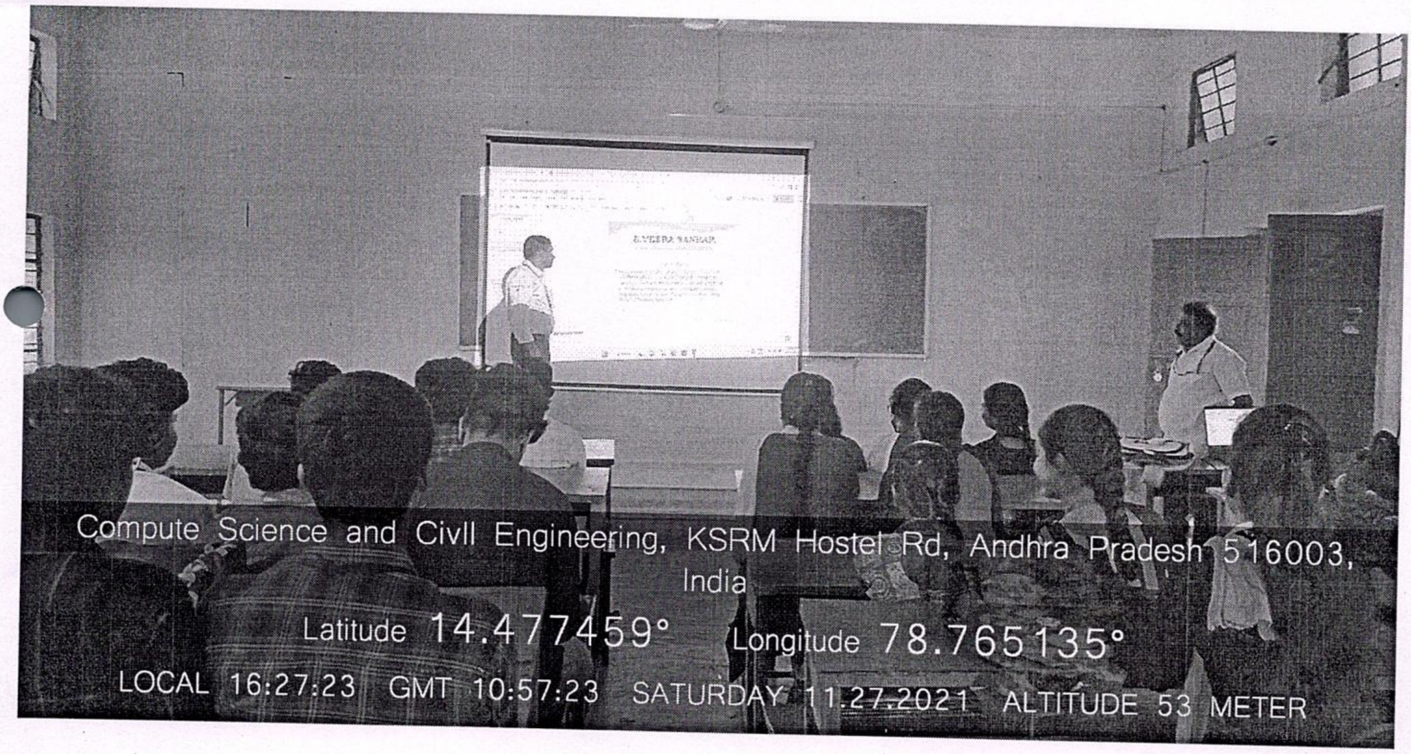

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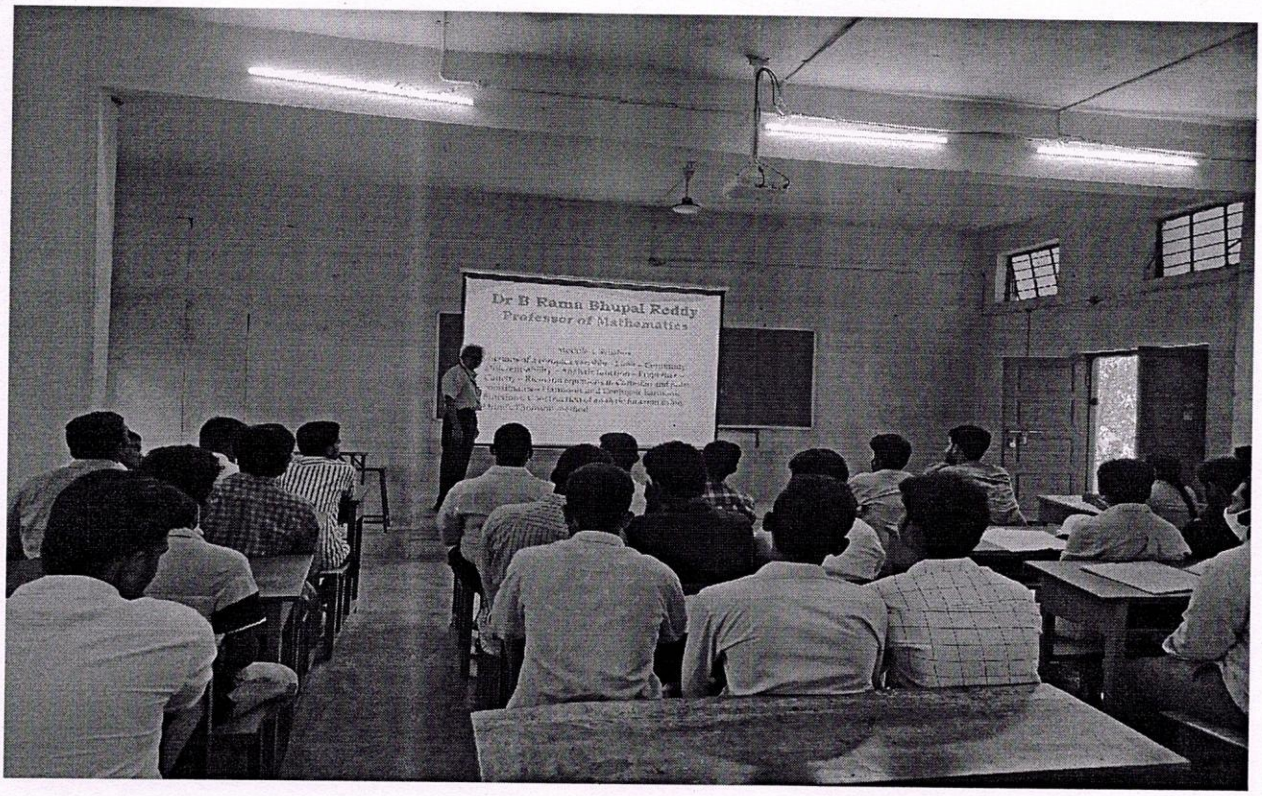
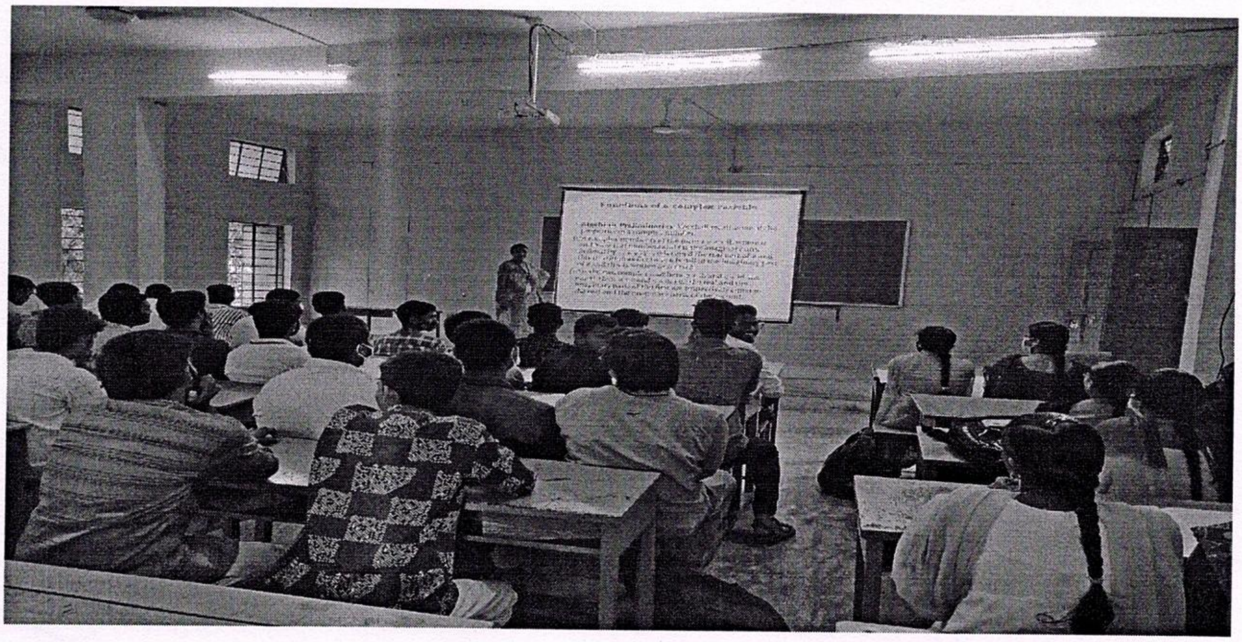
Photos





B. Ram

Coordinator

HODH&S

Dr. I. SREEVANI M.Sc.,Ph.D.
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This is to certify that M jaganmohan has successfully completed his certification course on Complex Analysis organized by Department of H&S, K.S.R.M.C.E, Kadapa, A.P from 26/10/2021 to 02/12/2021.

Sreevani
V.S.S. Murty

Dr. B.Rama Bhupal Reddy
Coordinator

Dr.I.Sreevani
HoD/H&S

Dr. V.S.S.Murthy
Principal

Prof.A.Mohan

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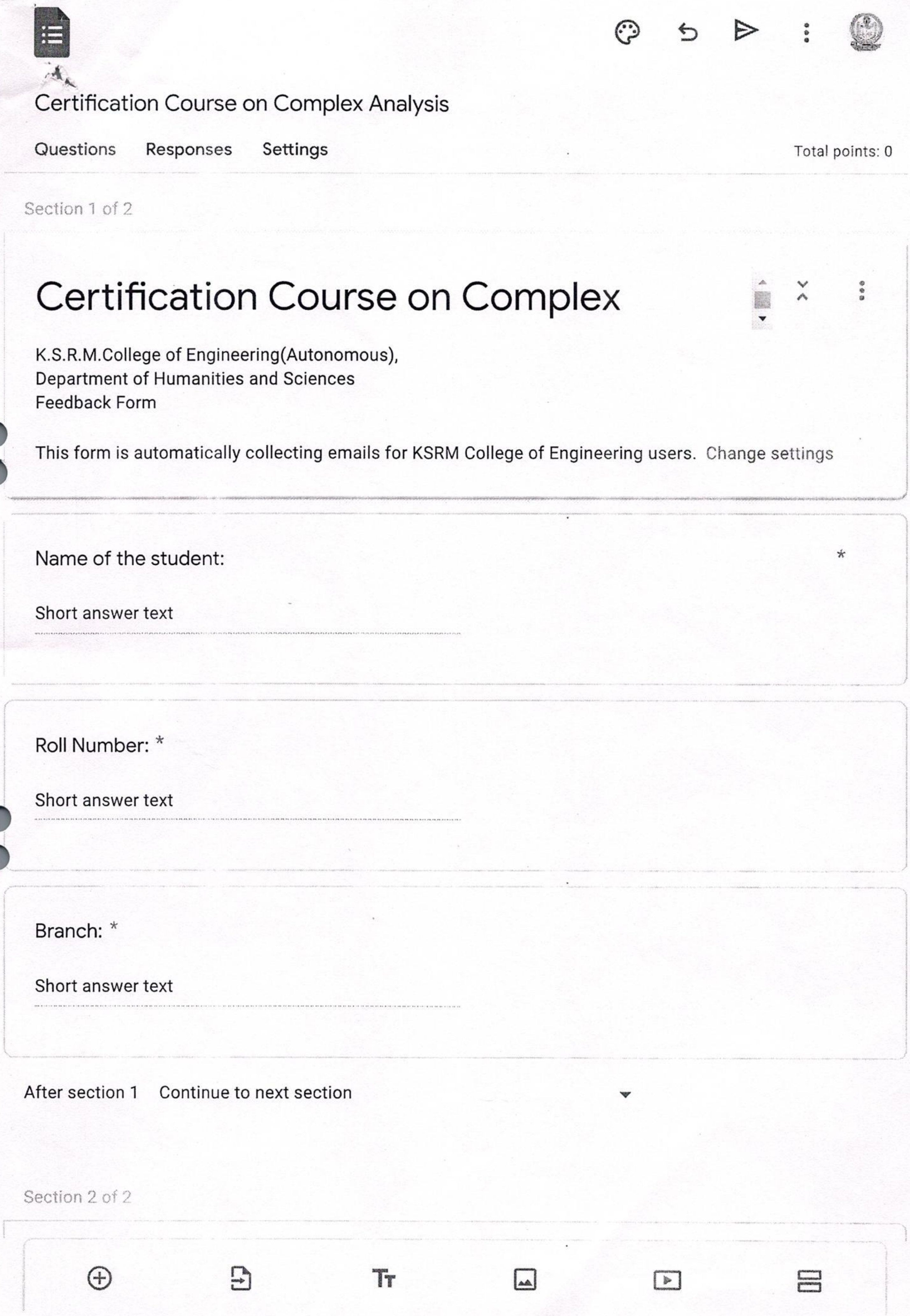
Sreevani
V.S.S. Murthy

Dr. B.Rama Bhupal Reddy
Coordinator

Dr.I.Sreevani
HoD/H&S

Dr. V.S.S.Murthy
Principal

Prof.A.Mohan



Certification Course on Complex Analysis

Section 1 of 2

Certification Course on Complex

K.S.R.M.College of Engineering(Autonomous),
Department of Humanities and Sciences
Feedback Form

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Name of the student: *

Short answer text

Roll Number: *

Short answer text

Branch: *

Short answer text

After section 1 Continue to next section

Section 2 of 2

Certification Course on Complex Analysis

1. Is the course content met your expectation *

○ Yes

○ Some What

○ Can not

○ Try

2. Rate your knowledge of the speakers in providing you the expected outcome *

	1	2	3	4	5	
Poor	○	○	○	○	○	Excellent

3. Is the lecture sequence well planned *

○ Yes

○ somewhat

○ Maybe

○ Cannot

4. The contents of the course is explained with clearly and examples *

○ Strongly Agree

○ Agree

○ Disagree

5. Please set your level of satisfaction for the curriculum set for the course *

○ Very satisfied

○ Satisfied

○ Neutral

○ Dissatisfied

Overall Opinion on the course *

Short answer text

Timestamp	Email Address	Full Name	Roll Number	Branc	1. Is the course	2. Rate your	3. Is the	4. The contents of the course is	5. Please set you	Overall Opinion on the course
12/2/2021 15:19:16	199y1a0109@ksrmce.a	DIRASANTHA CHENANKE	199y1a0109	CIVIL	Yes	5	Yes	Strongly Agree	Very satisfied	Use full for us
12/2/2021 15:11:08	199y1a0110@ksrmce.a	Gajula Mahamad Javid	199y1a0110	Civil	Yes	5	Yes	Agree	Satisfied	Good
12/2/2021 15:12:41	199y1a0133@ksrmce.a	P.Venkata Siva	199Y1A0133	Civil	Yes	5	Yes	Agree	Very satisfied	We learn Very Well
12/2/2021 15:07:24	199y1a0133@ksrmce.a	PAGIDI VENKATA SIVA	199Y1A0133	Civil	Yes	5	Yes	Agree	Satisfied	Course.We learn very
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12/2/2021 15:09:19	199y1a0137@ksrmce.a	P.Praveen kumar	199Y1A0137	Civil	Yes	5	Yes	Agree	Satisfied	It is good
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12/2/2021 15:19:59	209y5a0125@ksrmce.a	I.CHINNA PRASAD	209Y5A0125	CIVIL	Yes	3	Yes	Agree	Satisfied	Ok

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Module 1

Function of a Complex Variables:

If $z = x + iy$ and $w = u + iv$ are two complex variables, and if for each value of z in a certain portion of the complex plane (called also as the domain R of the complex plane) there corresponds one or more values of y, then w is said to be a function of z and is written as

$$w = f(z) = f(x + iy) = u(x, y) + i\, v(x, y) \qquad (1)$$

where $u(x, y)$and $v(x, y)$ are real functions of the real variables x and y. Clearly for a given value of z, the values of x and y are known and thus, one or more values of w are determined by (1). If for each value of z in R, there is correspondingly only one value of w, then w is called a *single-valued function* of z. If there is more than one value of w corresponding to a given value of z, then w is called a *multiple-valued function or many- valued function.*

For example, $w = z^2, w = \frac{1}{z}, w = \frac{z}{z^4+1}$ are single valued function of z. The function $w = z^{1/2}$, $w = \arg(z)$ are examples of many valued functions. The first one has three values for each value of z (except for $z = 0$) and the second one assumes infinite set of real values for each value of z other than $z = 0$.

The complex quantities z and w can be represented on separate complex planes, called the z-plane and the w-plane respectively. The relation $w = f(z)$ establishes correspondence between the points (x, y) of the z-plane and the points (u, v) of the w-plane.

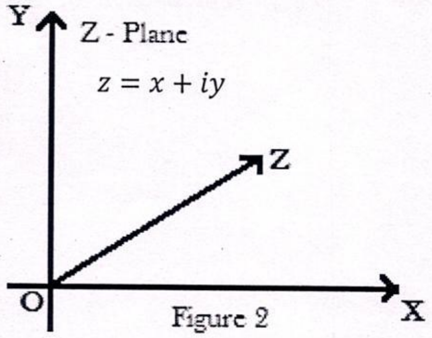


Figure 2

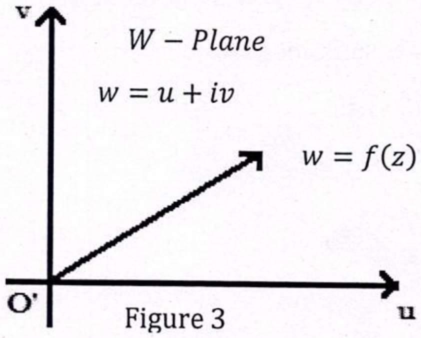


Figure 3

Limits: Let $w = f(z)$ denote some functional relationship connecting w with z.

Then $w = f(x + iy) = u(x, y) + i\, v(x, y)$ where u and v are real functions of x and y. As z *approaches* z_0, the limit of $f(z)$ is said to be w_0 if $f(z)$ can be kept arbitrarily close to w_0, by keeping z sufficiently close to, but different from z_0.

$$i.e., \lim_{z \to z_0} w = \lim_{z \to z_0} f(z) = w_0$$

Now let $z_0 = x_0 + iy_0$

when z approaches z_0, it means that $x \to x_0$ and $y \to y_0$.

Hence $\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to z_0} (u + i\, v) = \lim\limits_{\substack{x \to x_0 \\ y \to y_0}} (u + i\, v) = u_0 + i\, v_0$

Hence $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0$ and $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0$.

Note: In the above, when we say that $z \to z_0$, it means that $x \to x_0$ *and* $y \to y_0$ in any order, by any path as shown in figure 4.

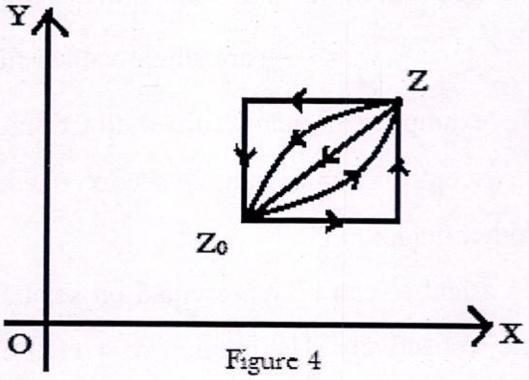


Figure 4

Continuity: The idea of continuity is closely connected with the concept of a limit. A single-valued function $w = f(z)$ is said to be continuous at a point $z = z_0$ provided each of the following conditions is satisfied:

(i) $f(z_0)$ exists

(ii) $\lim\limits_{z \to z_0} f(z)$ exists, and

(iii) $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Remarks:

1. If $f(z)$ is continuous at every point of a region R, it is said to be continuous throughout R.

2. $w = f(z) = u(x, y) + i\, v(x, y)$. If $f(z)$is continuous at $z = z_0$, then its real and imaginary parts, i.e., u and v will be continuous functions at $z = z_0$, i.e., at $x = x_0$ *and* $y =$

y_0. Conversely, if u and v are continuous functions at $z = z_0$, then $f(z)$ will be continuous at $z = z_0$.

3. The sums, differences and products of continuous functions are also continuous are also continuous. The quotient of two continuous functions is continuous except for those values of z for which the denominator vanishes.

Continuity of a Function of Two Real Variables:

$$w = f(z) = f(x + iy)$$

is a function of the two variables x and y. Hence, to discuss the continuity of $f(z)$, we shall have to deal with the continuity of a function of two independent variables x and y.

Definition: a function $f(x,\ y)$ of two real independent variables x and y is said to be continuous at a point $(x_0,\ y_0)$ if,

(i) $f(x_0,\ y_0)$, the value of $f(x,\ y)$ at $(x_0,\ y_0)$ is finite, and

(ii) $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0,\ y_0)$ in whatever way $x \to x_0$ and $y \to y_0$

To illustrate the idea of continuity of a function of two variables given in the following examples:

EX. 1. Show that $f(x,y) = \frac{2xy}{x^2+y^2}$ is discontinuous at origin, given that $f(0,0) = 0$.

Solution: Given $f(x,y) = \frac{2xy}{x^2+y^2}$

If $y \to 0$ first and then $x \to 0$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0}\frac{2xy}{x^2 + y^2} = \lim_{x\to 0}\frac{2x(0)}{x^2} = 0$$

If $x \to 0$ first and then $y \to 0$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0}\frac{2xy}{x^2 + y^2} = \lim_{y\to 0}\frac{2y(0)}{y^2} = 0$$

Let x and y both tend to zero simultaneously along the path $y = mx$.

Then, $\lim\limits_{\substack{y=mx\\ x\to 0}} f(x,y) = \lim\limits_{\substack{y=mx\\ x\to 0}}\frac{2xy}{x^2 + y^2} = \lim\limits_{x\to 0}\frac{2x.mx}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}$

This limit changes its value for different values of m.

when $m = 1, \frac{2m}{1+m^2} = 1$ and for $m = 2, \frac{2m}{1+m^2} = \frac{4}{5}$ and so on.

Hence $\lim\limits_{y\to 0}\frac{2xy}{x^2+y^2} \neq 0$, when $x \to 0, y \to 0$ in any manner. So the function is not continuous at the origin.

Derivative of a Function of a Complex Variable: For a real function of a single real variable say, $y = f(x)$, the derivative of y with Respect to x is defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Hence Δx can approach zero in only one way.

Let $w = f(z)$ be a single-valued function of z. Then, the derivative of w is defined to be

$$\frac{dw}{dz} = f^{'}(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided the above limit exists and is the same, in whatever manner Δz approaches zero.

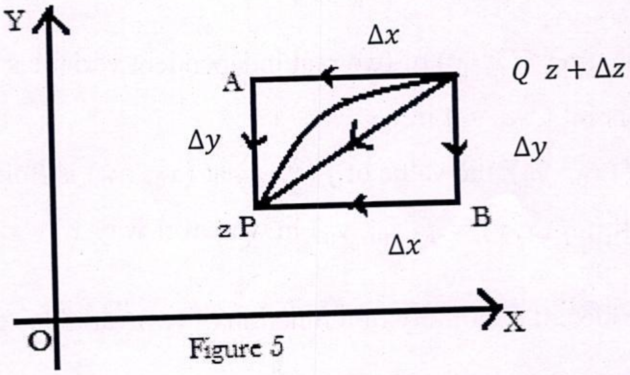


Figure 5

We can show by a figure that Δz can approach zero in several ways. P is the point in the z-plane corresponding to $z = x + i\,y$. Q is the point $z + \Delta z$. $\Delta z = \Delta x + i\,\Delta y$, where $\Delta x, \Delta y$ are small increments in x and y respectively. As $\Delta z \to 0,\ i.e., \Delta x, \Delta y also \to 0$ and the point Q approaches to P. Now Q can approach P along the rectilinear path QAP on which first Δx and then Δy approach zero or Q may approach P along the rectilinear path QBP on which first Δy and then Δx approach zero. More generally, Q can approach P along infinitely many paths, $i.e.,$ Δz approaches zero in several ways.

Hence, in the definition of $f^{'}(z)$, the derivative of $f(z)$, it is necessary that the limit of the difference quotient

$$i.e., \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

should be the same, no matter how Δz approaches zero. When this limit is unique, the function is said to be differentiable. This severe restriction narrows down greatly the class of functions of a complex variable that possess derivatives.

Thus we find that $\frac{dw}{dz}$ depends not only upon z but also upon the manner in which Δz approaches zero. To illustrate this, consider the simple case,

$$w = f(z) = x - i\,y$$

Then

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{[(x+\Delta x) - i\,(y+\Delta y)] - (x - i\,y)}{\Delta x + i\,\Delta y}$$

$$= \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y}$$

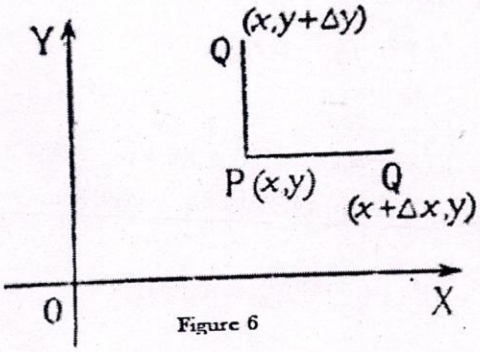


Figure 6

Now, let $\Delta z \to 0$ is such a way that first Δy and then Δx approach zero, i.e., Q approaches P along the horizontal line. Then

$$\lim_{\Delta z \to 0} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

But, suppose Q approaches P along the vertical line so that first Δx and then Δy approach zero. Then

$$\lim_{\Delta z \to 0} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} = \lim_{\Delta y \to 0} \frac{-i\,\Delta y}{i\,\Delta y} = -1$$

For other paths of approach of Q towards P, we can get as many distinct values of the above limit as we please. We therefore say that $f(z) = x - i\,y$ possesses no derivative.

Definition: If a single-valued function $w = f(z)$ possesses a derivative at $z = z_0$ and at every point in some neighbourhood of z_0, then $f(z)$ is said to be *analytic* at z_0 and z_0 is called a *regular point* of the function. If $f(z)$ is analytic at every point of a region R, then we say that $f(z)$is analytic in R. A point at which an analytic function ceases to have a derivative is called a singular point. An analytic function is also referred to as *regular* or *holomorphic*.

Conditions under which $w = f(z)$is analytic:

Let $w = f(z)$ be an analytic function of a complex variable in a region R. Then $f'(z)$ exists at every point in R. Let us now find the conditions for the existence of the derivative of $f(z)$ at a point z.

Let $z = x + i\,y$ and $w = f(z) = f(x + i\,y) = u(x,y) + i\,v(x,y)$ where u and v are functions of x and y. Let Δx and Δy be the increments in x and y respectively and let Δz be the corresponding increment in z

Then $z + \Delta z = (x + \Delta x) + i(y + \Delta y)$

Hence $\Delta z = \Delta x + i\,\Delta y$

Also $f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + i\,v(x + \Delta x, y + \Delta y)$

Hence $\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{[u(x+\Delta x,y+\Delta y)+i\,v(x+\Delta x,y+\Delta y)]-[u(x,y)+i\,v(x,y)]}{\Delta x+i\,\Delta y}$

As $\Delta z \to 0$, we have $\Delta x \to 0$ and $\Delta y \to 0$.

Hence by definition,

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{[u(x + \Delta x, y + \Delta y) + i\,v(x + \Delta x, y + \Delta y)] - [u(x,y) + i\,v(x,y)]}{\Delta x + i\,\Delta y} \qquad (1)$$

If $f(z)$ is analytic, $f'(z)$ must have a unique value, in whatever manner $\Delta z \to 0$. Now let $\Delta z \to 0$ in such a way that first Δy and then $\Delta x \to 0$. Then from (1),

$$f'(z) = \lim_{\Delta x\to 0} \frac{[u(x + \Delta x, y) + i\,v(x + \Delta x, y)] - [u(x,y) + i\,v(x,y)]}{\Delta x}$$

$$i.e., f'(z) = \lim_{\Delta x\to 0} \frac{[u(x + \Delta x, y) - u(x,y)] + i\,[v(x + \Delta x, y) - v(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{u(x + \Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x + \Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (2)$$

(by definition of partial derivatives)

Since $f'(z)$ is to be unique, it is necessary that the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ must exists at the point (x,y).

Secondly, let $\Delta z \to 0$ such that $\Delta x \to 0$ first and then $\Delta y \to 0$. Then from (1)

$$f'(z) = \lim_{\Delta y\to 0} \frac{[u(x, y + \Delta y) + i\,v(x, y + \Delta y)] - [u(x,y) + i\,v(x,y)]}{i\,\Delta y}$$

$$i.e., f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y + \Delta y) - u(x, y)] + i\,[v(x, y + \Delta y) - v(x, y)]}{i\,\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\,\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y} \qquad (3)$$

Hence $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ must exist at (x, y).

Now, if the derivative $f'(z)$ exists, it is necessary that the two expressions (2) and (3) which we have derived for it must be the same. Hence equating these expressions, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (4)$$

$$\text{and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad (5)$$

$$i.e., u_x = v_y \text{ } and \text{ } v_x = -u_y$$

The equations (4) and (5) are called ***Cauchy-Riemann differential equations***.

Note: The Cauchy-Riemann equations are only the necessary conditions for the function $f(z) = u + i\,v$ to be differentiable i.e., if the function is differentiable, then it must satisfy these equations. But the converse is not necessarily true. A function may satisfy these equations at a point and yet it may not be differentiable at that point.

Hence the conditions expressed by Cauchy-Riemann equations (C-R equations) are only *necessary but not sufficient* for a function to be analytic.

Sufficient Conditions for $f(z)$ to be Analytic: We shall now prove the following theorem

The single valued continuous function $w = f(z) = u + i\,v$ analytic in a region R, if the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ *and* $\frac{\partial v}{\partial y}$ exist, are continuous and satisfy the ***Cauchy-Riemann equations*** at each point in$R$.

Proof: Let $w = f(z) = u(x, y) + i\,v(x, y)$

It is now given that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ and\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \qquad (1)$$

Also these partial derivatives are continuous.

Then $\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$

$= [u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y)] + [u(x+\Delta x, y) - u(x,y)]$

$= \Delta y.\frac{\partial}{\partial y}u(x+\Delta x, y+\theta_1.\Delta y) + \Delta x.\frac{\partial}{\partial x}u(x+\theta_2.\Delta x, y)$

Using the first Mean Value Theorem, θ_1 and θ_2 being both positive and less than 1.

Now, at the point (x,y) the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous.

Hence the above expression Δu may be written as

$$\Delta u = \Delta x.\left[\frac{\partial u}{\partial x}+\lambda_1\right] + \Delta y.\left[\frac{\partial u}{\partial y}+\lambda_2\right] \qquad (2)$$

where λ_1 and λ_2 both tend to zero as $|\Delta z| \to 0$

Similarly, using the result that the derivatives $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous, we get

$$\Delta v = \Delta x.\left[\frac{\partial v}{\partial x}+\mu_1\right] + \Delta y.\left[\frac{\partial v}{\partial y}+\mu_2\right] \qquad (3)$$

where μ_1 and μ_2 both tend to zero as $|\Delta z| \to 0$

Now $\Delta w = \Delta u + i\,\Delta v$

$$= \left\{\Delta x.\left[\frac{\partial u}{\partial x}+\lambda_1\right] + \Delta y.\left[\frac{\partial u}{\partial y}+\lambda_2\right]\right\} + i\left\{\Delta x.\left[\frac{\partial v}{\partial x}+\mu_1\right] + \Delta y.\left[\frac{\partial v}{\partial y}+\mu_2\right]\right\}$$

$$= \Delta x\left(\frac{\partial u}{\partial x}+i\,\frac{\partial v}{\partial x}\right) + \Delta y\left(\frac{\partial u}{\partial y}+i\,\frac{\partial v}{\partial y}\right) + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y \qquad (4)$$

where $\varepsilon_1 = \lambda_1 + i\,\mu_1$ and $\varepsilon_2 = \lambda_2 + i\,\mu_2$ and $\varepsilon_{1,}\varepsilon_2 \to 0$ as $|\Delta z| \to 0$.

In (4), apply the conditions (1) *i.e.*, put

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Then $\Delta w = \Delta x\left(\frac{\partial u}{\partial x}+i\,\frac{\partial v}{\partial x}\right) + \Delta y\left(-\frac{\partial v}{\partial x}+i\,\frac{\partial u}{\partial x}\right) + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$

$= (\Delta x + i\,\Delta y)\frac{\partial u}{\partial x} + i\,(\Delta x + i\,\Delta y)\frac{\partial v}{\partial x} + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$

$= (\Delta x + i\,\Delta y)\left[\frac{\partial u}{\partial x}+i\,\frac{\partial v}{\partial x}\right] + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$

Hence $\quad \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x}+i\,\frac{\partial v}{\partial x}+\varepsilon_1\frac{\Delta x}{\Delta z}+\varepsilon_2\frac{\Delta y}{\Delta z} \qquad (5)$

By definition, $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[u(r+\Delta r, \theta+\Delta\theta) + i\, v(r+\Delta r, \theta+\Delta\theta)] - [u(r,\theta) + i\, v(r,\theta)]}{\Delta(r\, e^{i\theta})} \qquad (1)$$

If $f(z)$ is analytic, $f'(z)$ must have a unique value in whatever manner $\Delta z \to 0$.

First let $\Delta z \to 0$ along a radius vector through the origin.

i.e., keep θ constant.

Then $\Delta z = \Delta(r\, e^{i\theta}) = e^{i\theta}\Delta r.$

As $\Delta z \to 0$, $\Delta r \to 0$. So (1) gives

$$f'(z) = \lim_{\Delta r \to 0} \frac{[u(r+\Delta r, \theta) + i\, v(r+\Delta r, \theta)] - [u(r,\theta) + i\, v(r,\theta)]}{e^{i\theta}\Delta r}$$

$$= e^{-i\theta} \lim_{\Delta r \to 0} \left[\frac{u(r+\Delta r, \theta) - u(r,\theta)}{\Delta r} + i\, \frac{v(r+\Delta r, \theta) - v(r,\theta)}{\Delta r}\right]$$

$$= e^{-i\theta} \left[\lim_{\Delta r \to 0} \frac{u(r+\Delta r, \theta) - u(r,\theta)}{\Delta r} + i \lim_{\Delta r \to 0} \frac{v(r+\Delta r, \theta) - v(r,\theta)}{\Delta r}\right]$$

$$= e^{-i\theta} \left(\frac{\partial u}{\partial r} + i\, \frac{\partial v}{\partial r}\right) \qquad (2)$$

Secondly, keep r constant.

Then $\Delta z = \Delta(r\, e^{i\theta}) = ire^{i\theta}\Delta\theta$

As $\Delta z \to 0$, $\Delta\theta \to 0$. So (1) gives

$$f'(z) = \lim_{\Delta\theta \to 0} \frac{[u(r, \theta+\Delta\theta) + i\, v(r, \theta+\Delta\theta)] - [u(r,\theta) + i\, v(r,\theta)]}{ire^{i\theta}\Delta\theta}$$

$$= \frac{1}{re^{i\theta}} \lim_{\Delta\theta \to 0} \frac{[u(r, \theta+\Delta\theta) + i\, v(r, \theta+\Delta\theta)] - [u(r,\theta) + i\, v(r,\theta)]}{i\Delta\theta}$$

$$= \frac{1}{re^{i\theta}} \lim_{\Delta\theta \to 0} \frac{[u(r, \theta+\Delta\theta) - u(r,\theta)] + i[\, v(r, \theta+\Delta\theta) - \, v(r,\theta)]}{i\Delta\theta}$$

$$= \frac{1}{re^{i\theta}} \left[-i \lim_{\Delta\theta \to 0} \frac{u(r, \theta+\Delta\theta) - u(r,\theta)}{\Delta\theta} + \lim_{\Delta\theta \to 0} \frac{v(r, \theta+\Delta\theta) - v(r,\theta)}{\Delta\theta}\right]$$

$$= \frac{1}{r} e^{-i\theta} \left(-i\, \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right) \qquad (3)$$

Now $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$

and so $\left|\frac{\Delta x}{\Delta z}\right| \leq 1$ and $\left|\frac{\Delta y}{\Delta z}\right| \leq 1$.

Also $\varepsilon_{1,}\varepsilon_2 \to 0$ as $|\Delta z| \to 0$

So proceeding to the limit as $\Delta z \to 0$, (5) gives

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$$i.e., f'(z) \text{ exists and is equal to } \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

We shall put the above discussion in 4.7 and 4.8 relating to differentiability in the form of a theorem as follows.

If u and v are real single-valued functions of x and y which, with their four first order partial derivatives$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y}\right)$, are continuous throughout a region R, then the *Cauchy-Riemann equations*

$$u_x = v_y \; and \; v_x = -u_y$$

are both necessary and sufficient condition, so that $f(z) = u + i\,v$ may be analytic. The derivative of $f(z)$ is then given by either of the expressions

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \text{ or } f'(z) = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

Derive the *Cauchy-Riemann equations* if $f(z)$ is expressed in polar coordinates.

Solution: Let $f(z) = u(r,\theta) + i\,v(r,\theta)$ in polar coordinates.

$z = x + i\,y = r(cos\theta + i\,sin\theta) = r\,e^{i\theta}$.

Let $\Delta r\; and\; \Delta\theta$ be the increments in $r\; and\; \theta$ respectively and let Δz be the corresponding increment in z.

$$\Delta z = \Delta\left(r\,e^{i\theta}\right)$$

$$f(z + \Delta z) = u(r + \Delta r, \theta + \Delta\theta) + i\,v(r + \Delta r, \theta + \Delta\theta)$$

$$f(z + \Delta z) - f(z) = [u(r + \Delta r, \theta + \Delta\theta) + i\,v(r + \Delta r, \theta + \Delta\theta)] - [u(r,\theta) + i\,v(r,\theta)]$$

Hence $\dfrac{f(z + \Delta z) - f(z)}{\Delta z}$

$$= \frac{[u(r + \Delta r, \theta + \Delta\theta) + i\,v(r + \Delta r, \theta + \Delta\theta)] - [u(r,\theta) + i\,v(r,\theta)]}{\Delta z}$$

$$= \frac{[u(r + \Delta r, \theta + \Delta\theta) + i\,v(r + \Delta r, \theta + \Delta\theta)] - [u(r,\theta) + i\,v(r,\theta)]}{\Delta(r\,e^{i\theta})}$$

Since $f(z)$ is analytic, $f'(z)$ must have a unique value in whatever manner $\Delta z \to 0$. Then From (2) and (3), we get

$$\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r} = \frac{1}{r}\left(-i\,\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right)$$

Equating on both sides real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad (4)$$

$$\text{and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad (5)$$

These equations are the ***Cauchy-Riemann equations*** if $f(z)$ is expressed in polar coordinates.

Note: Differentiating (4) partially with respect to r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial \theta} \qquad (6)$$

Differentiating (5) partially with respect to θ, we get

$$\frac{\partial^2 u}{\partial \theta^2} = -r\,\frac{\partial^2 v}{\partial \theta\,\partial r} \qquad (7)$$

Thus using (4), (6) and (7), we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \left(\text{since} \frac{\partial^2 v}{\partial r\,\partial \theta} = \frac{\partial^2 v}{\partial \theta\,\partial r}\right)$$

EX. 3. Show that $w = f(z) = \bar{z} = x - i\,y$ is not analytic anywhere in the complex plane.

Solution: Let $w = u + i\,v = x - i\,y.$

Here $u = x$ and $v = -y$

Then $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0 \; and \; \frac{\partial v}{\partial y} = -1$

Hence $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ but $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

The second of the Cauchy-Riemann equations is satisfied everywhere, but not so the first. So $w = \bar{z}$ is not analytic anywhere in the complex plane.

EX. 4. Show that $w = f(z) = z = x + i\,y$ is analytic anywhere in the complex plane.

Solution: Let $w = u + i\,v = x + i\,y.$

Here $u = x$ and $v = y$

Then $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0 \; and \; \frac{\partial v}{\partial y} = 1$

Differentiation Formulas: We have already defined the derivative of $w = f(z)$ to be

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

This definition is identical in form to that of the derivative of a function of a real variable. Hence the fundamental formulas for differentiation in the domain of complex numbers are the same as those in the case of real variables. Thus we have the following formulas:

(i) If k is a complex constant, then $\frac{d}{dz}(k) = 0$.

(ii) If k is a complex constant and w is a differentiable function, $\frac{d}{dz}(kw) = k\frac{dw}{dz}$.

(iii) If $w_1(z)$ and $w_2(z)$ are two differentiable functions, then $\frac{d}{dz}(w_1 \mp w_2) = \frac{dw_1}{dz} \mp \frac{dw_2}{dz}$.

(iv) $\frac{d}{dz}(w_1 . w_2) = w_1 . \frac{dw_2}{dz} + w_2 . \frac{dw_1}{dz}$

(v) $\frac{d}{dz}\left(\frac{w_1}{w_2}\right) = \frac{w_2 . \frac{dw_1}{dz} - w_1 . \frac{dw_2}{dz}}{w_2^2}$

(vi) If w is a function of $w_1(z)$, $\frac{dw}{dz} = \frac{dw}{dw_1} . \frac{dw_1}{dz}$

(vii) If n is a positive integer, $\frac{d}{dz}(z^n) = n . z^{n-1}$. This can be extended to the case when n is a negative integer or any fraction.

EX. Find where the function $w = f(z) = \frac{1}{z}$ ceases to be analytic.

Solution: Given that $w = f(z) = \frac{1}{z}$

$$\frac{dw}{dz} = \frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2} \; if \; z \neq 0$$

For $z = 0$, $\frac{dw}{dz}$ does not exist. So, w is analytic everywhere except at the point $z = 0$ which is singular point of $f(z)$.

EX. Show that an analytic function with constant real part is constant and an analytic function with constant modulus is also constant.

Solution: Let $w = f(z) = u + i\,v$ be an analytic function.

(a) Let $u(x, y) = a \; constant = c_1$

Since $f(z)$ is analytic, $f^{'}(z)$ must have a unique value in whatever manner $\Delta z \to 0$. Then From (2) and (3), we get

$$\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r} = \frac{1}{r}\left(-i\,\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right)$$

Equating on both sides real and imaginary parts, we get

θ

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad (4)$$

$$\text{and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad (5)$$

These equations are the ***Cauchy-Riemann equations*** if $f(z)$ is expressed in polar coordinates.

EX. Show that $w = f(z) = \bar{z} = x - i\,y$ is not analytic anywhere in the complex plane.

Solution: Let $w = u + i\,v = x - i\,y$.

Here $u = x$ and $v = -y$

Then $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0 \; and \; \frac{\partial v}{\partial y} = -1$

Hence $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ but $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

The second of the Cauchy-Riemann equations is satisfied everywhere, but not so the first. So $w = \bar{z}$ is not analytic anywhere in the complex plane.

EX. Show that $w = f(z) = z = x + i\,y$ is analytic anywhere in the complex plane.

Solution: Let $w = u + i\,v = x + i\,y$.

Here $u = x$ and $v = y$

Then $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0 \; and \; \frac{\partial v}{\partial y} = 1$

Hence $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ at all points in the complex plane. The C-R equations are identically satisfied. Further these four partial derivatives are continuous.

Hence $w = f(z) = z$is analytic anywhere in the complex plane.

EX. Prove that the function $f(z)$ where

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, when\ z\neq 0, f(0)=0$$

is continuous at $z=0$. Prove also that the C-R equations are satisfied by $f(z)$ at $z=0$ and yet $f'(z)$ does not exist at $z=0$.

Solution: Given that $f(z)=\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, when\ z\neq 0$

$$\lim_{z\to 0} f(z)=\lim_{\substack{y\to 0\\ x\to 0}}\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$=\lim_{x\to 0}\frac{x^3(1+i)}{x^2}=0$$

$$\lim_{z\to 0} f(z)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$=\lim_{y\to 0}\frac{-y^3(1-i)}{y^2}=0$$

Also $f(0)=0$ be given.

Hence

$$\lim_{z\to 0} f(z)=f(0)$$

When $x\to 0$ first and then $y\to 0$ and also When $y\to 0$ first and then $\to 0$.

Let $x\ and\ y$ both tend to zero simultaneously along the path $y=mx^n$.

For $n=1$, this is a straight line and for $n=2,3,\ldots$, we will get different curves passing through the points (x,y) and the origin. Then

$$\lim_{z\to 0} f(z)=\lim_{\substack{y=mx^n\\ x\to 0}}\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$=\lim_{x\to 0}\frac{x^3(1+i)-(mx^n)^3(1-i)}{x^2+(mx^n)^2}$$

$$=\lim_{x\to 0}\frac{x^3[1+i-m^3x^{3n-3}(1-i)]}{x^2[1+m^2x^{2n-2}]}$$

$$=\lim_{x\to 0}\frac{x[1+i-m^3x^{3n-3}(1-i)]}{1+m^2x^{2n-2}}$$

$$=\lim_{x\to 0}\frac{x[1+i-m^3(x^{n-1})^3(1-i)]}{1+m^2(x^{n-1})^2}=0$$

(because when $n>1, n-1$ is positive and $\lim\limits_{x\to 0}x^{n-1}=0$)

When $n=1$ the above limit

$$= \lim_{x\to 0} \frac{x[1+i-m^3(1-i)]}{1+m^2} = 0$$

Hence $\lim_{z\to 0} f(z) = f(0)$ in whatever manner $z \to 0$.

Therefore $f(z)$ is continuous at the origin.

Now $f(z) = \frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2} = u(x,y) + i\,v(x,y)$

Here $u(x,y) = \frac{x^3-y^3}{x^2+y^2}$ and $v(x,y) = \frac{x^3+y^3}{x^2+y^2}$

Since $f(0) = 0, u(0,0) = 0\ and\ v(0,0) = 0$.

Now at origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x^3}{x^3} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{-y^3}{y^3} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x^3}{x^3} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{y^3}{y^3} = 1$$

Hence at origin,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ and\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

So the C-R equations are satisfied at the origin.

Now, by the definition

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{f(z)}{z}$$

$$= \lim_{z\to 0} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)\,(x+i\,y)}$$

Let $y \to 0$ first and then $x \to 0$.

$$f^{'}(0) = \lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)\,(x+i\,y)}$$

$$= \lim_{x\to 0} \frac{x^3(1+i)}{x^3} = 1+i$$

If $x \to 0$ first and then $y \to 0$.

$$f^{'}(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)\,(x+i\,y)}$$

$$= \lim_{y\to 0} \frac{-y^3(1-i)}{i\,y^3} = \frac{-(1-i)}{i} = i+1$$

Generally when $z \to 0$ along the path $y = mx$,

$$f^{'}(0) = \lim_{\substack{y=mx\\ x\to 0}} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)\,(x+i\,y)}$$

$$= \lim_{x\to 0} \frac{x^3(1+i) - (mx)^3(1-i)}{(x^2+m^2x^2)\,(x+i\,mx)}$$

$$= \frac{(1+i) - m^3(1-i)}{(1+m^2)\,(1+i\,m)}$$

This assumes different values, as m varies, $f^{'}(z)$ has no unique value at origin, i.e., $f(z)$ is not differentiable at that point.

Hence we find that even at a point, if $f(z)$ is continuous and satisfies the C-R equations, the function need not be differentiable.

Differentiation Formulas: We have already defined the derivative of $w = f(z)$ to be

$$\frac{dw}{dz} = f^{'}(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

EX. Find where the function $w = f(z) = \frac{1}{z}$ ceases to be analytic.

Solution: Given that $w = f(z) = \frac{1}{z}$

$$\frac{dw}{dz} = \frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2} \; if\; z \neq 0$$

For $z = 0$, $\frac{dw}{dz}$ does not exist. So, w is analytic everywhere except at the point $z = 0$ which is singular point of $f(z)$.

Properties of Analytic Functions:

Property 1. Both the real part and the imaginary part of any analytic function satisfy Laplace's equation

$$\frac{\partial^2 \emptyset}{\partial x^2}+\frac{\partial^2 \emptyset}{\partial y^2}=0$$

Proof: Let $f(z)=u+i\,v$ be analytic in some domain of the z-plane.

Then u and v satisfy the C-R equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad (1)$$

$$\text{and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad (2)$$

Differentiating (1) with respect to x and (2) with respect to y partially, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \text{and} \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$\boldsymbol{i.e.,} \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x \partial y} \qquad (3)$$

$$\text{and } \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y \partial x} \qquad (4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x \partial y}-\frac{\partial^2 v}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (5)\left(\text{since } \frac{\partial^2 v}{\partial x \partial y}=\frac{\partial^2 v}{\partial y \partial x}\right)$$

Similarly we can show that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad (6)$$

(5) and (6) shows that u and v satsfy the Laplace's equation

$$\frac{\partial^2 \emptyset}{\partial x^2}+\frac{\partial^2 \emptyset}{\partial y^2}=0 \qquad (7)$$

which is Laplace's partial differential equation in the two independent variables x and y. This equation occurs frequently in mathematical physics. It is satisfied by the potential at a point not occupied by matter in a two-dimensional gravitational field. It is also satisfied by the velocity potential and stream function of two-dimensional irrotational flow of an incompressible non-viscous fluid.

Note: In proving results (5) and (6), it has assumed that the second order partial derivatives of u and v with respect to x and y all exist and further are continuous.

Any function which possesses continuous second order partial derivatives and which satisfies Laplace's equation is called a ***harmonic function***. Two harmonic functions, u and v which are such that $u + i\,v$ is an analytic function are called ***conjugate harmonic functions***.

The importance of analytic function of a complex variable is that such a function furnishes us with distinct solutions of Laplace's equation. It is this connection of analytic function with Laplace's equation that has given a great importance to the theory of functions of a complex variable in applied mathematics.

The equation (7) is written as $\nabla^2 \emptyset = 0$, where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

∇^2 is called the Laplacian operator.

Property 2. If $w = f(z) = u + i\,v$ is an analytic function, the curves of the family $u(x, y) = constant = c_1$ cut orthogonally the curves of the family $v(x, y) = constant = c_2$.

Proof: Given that $w = f(z) = u + i\,v$ is an analytic function

Then u and v are satisfy C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (2)$$

Suppose $u(x, y) = c_1$ is the equation of the family of curves for different values of c_1. Similarly, $v(x, y) = c_2$ is the equation of the family of curves for different values of c_2.

Since $u(x, y) = c_1$, by the total differentiation,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\text{Hence } \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} . \frac{dy}{dx} = 0 \text{ since } u = c_1$$

$$\text{So } \frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} \qquad (3)$$

This is the slope of the general curve of the u-family.

Similarly for the v –family,

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} \qquad (4)$$

Using (1) and (2), (4) can be written as

$$\frac{dy}{dx}=\frac{\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)}$$

The product of the slopes of the two families is

$$=-\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}\ X\ \frac{\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)}=-1.$$

Hence the curves cut each other orthogonally. The two families are said to be the ***orthogonal trajectories*** of one another.

Result 1. To prove that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial\bar{z}}$

Proof: Let $z=x+iy \text{ and } \bar{z}=x-iy$ so that

$$x=\frac{z+\bar{z}}{2}\ and\ y=\frac{z-\bar{z}}{2i}=-\frac{i(z-\bar{z})}{2}$$

This implies $\frac{\partial x}{\partial z}=\frac{1}{2}=\frac{\partial x}{\partial\bar{z}},\frac{\partial y}{\partial z}=-\frac{i}{2}=-\frac{\partial y}{\partial\bar{z}}$

Let $f=f(x,y)$. Then $f=f(z,\bar{z})$

We have

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$$

$$=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial\bar{z}}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\bar{z}}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\bar{z}}$$

$$=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)$$

Now $\frac{\partial^2 f}{\partial z\partial\bar{z}}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial\bar{z}}\right)$

$$=\frac{1}{4}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f$$

$$=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f$$

$$\therefore\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial\bar{z}}$$

Result 2. If $f(z)$ is a regular function of z; Prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Proof: Recall that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial \bar{z}}$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\partial \bar{z}}\, f(z)f(\bar{z})\ as\ |z|^2 = z\,\bar{z}$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\, f(z)f(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}[f(z)f'(\bar{z})]$$

$$= 4f'(z)f'(\bar{z}) = 4|f'(z)|^2$$

(since f(z) is treated as constant in differentiating with respect to $\bar{z}$)

Result 3. If $w = f(z)$ is a regular function of z such that $f'(z) \neq 0$. Prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)log\,|f'(z)| = 0$$

If $|f'(z)|$ is the product of a function of x and function of y, then show that $f'(z) = e^{\alpha z^2+\beta z+\gamma}$ where α is the real and β, γ are complex constants.

Proof: Recall that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial \bar{z}}$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)log\,|f'(z)| = 4\frac{\partial^2}{\partial z\partial \bar{z}}\,log\,|f'(z)|$$

$$= 2\frac{\partial^2}{\partial z\partial \bar{z}}\,log\,|f'(z)|^2$$

$$= 2\frac{\partial^2}{\partial z\partial \bar{z}}\,log\{f'(z)f'(\bar{z})\}\ as\ |z|^2 = z\,\bar{z}$$

$$= 2\frac{\partial^2}{\partial z\partial \bar{z}}\,log\{f'(z)\} + 2\frac{\partial^2}{\partial z\partial \bar{z}}\,log\{f'(\bar{z})\}$$

$$= 2\frac{\partial}{\partial \bar{z}}\left\{\frac{f''(z)}{f'(z)}\right\} + 2\frac{\partial}{\partial z}\left\{\frac{f''(\bar{z})}{f'(\bar{z})}\right\}$$

$$= 0 + 0 = 0$$

It follows from the fact that $f(z)$ is treated as constant in differentiation with respect to $\bar{z}$.

$$\text{Hence } \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)log\,|f'(z)| = 0 \qquad (1)$$

Let $|f^{'}(z)| = \emptyset(x).\psi(y)$

From (1),

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) log\ (\emptyset(x).\psi(y)) = 0$$

$$\frac{\partial^2}{\partial x^2}[log\ \emptyset(x) + log\ \psi(y)\ ] + \frac{\partial^2}{\partial y^2}[log\ \emptyset(x) + log\ \psi(y)\ ] = 0$$

$$\frac{\partial^2}{\partial x^2}[log\ \emptyset(x)] + \frac{\partial^2}{\partial y^2}[log\ \psi(y)\ ] = 0$$

$$\frac{d^2}{dx^2}[log\ \emptyset(x)] + \frac{d^2}{dy^2}[log\ \psi(y)\ ] = 0$$

$$\frac{d^2}{dx^2}[log\ \emptyset(x)] = -\frac{d^2}{dy^2}[log\ \psi(y)\ ] = 2p, say$$

For L.H.S. and R.H.S both are independent of each other.

$\frac{d^2}{dx^2}[log\ \emptyset(x)] = 2p$, given an integration

$$\frac{d}{dx}[log\ \emptyset(x)] = 2px + q$$

Again integrating, $log\ \emptyset(x) = px^2 + qx + r$

Similarly, $-log\ \psi(y) = py^2 + q_1y + r_1$

$$\begin{aligned} log\ (\emptyset(x).\psi(y)) &= log\ \emptyset(x) + log\ \psi(y) \\ &= px^2 + qx + r - py^2 - q_1y - r_1 \\ &= p(\, x^2 - y^2) + (qx - q_1y) + (r - r_1) \end{aligned}$$

or $\quad |f^{'}(z)| = \emptyset(x).\psi(y)$

$$= exp\ [p(\, x^2 - y^2) + (qx - q_1y) + (r - r_1)] \qquad (2)$$

Now $|exp(\alpha z^2 + \beta z + \gamma)| = |exp\{\alpha(x+iy)^2 + \beta(x+iy) + \gamma\}|$

$$= |exp\{[\alpha(x^2 - y^2) + 2i\alpha xy] + (a+ib)(x+iy) + (c+id)\}|$$

as α is a real.

$$= |exp\{\alpha(x^2 - y^2) + ax - by + c\} + exp\{i(2\alpha xy + bx + ay + d)\}|$$

$$= |exp\{\alpha(x^2 - y^2) + ax - by + c\}|$$

As $|e^{ip}| = 1$ for any real p, which of the same form as (2).

Hence we can write

$$f^{'}(z) = \ exp(\alpha z^2 + \beta z + \gamma)$$

Result 4. If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$$

Proof: Let $f(z) = u + iv$, then $Rf(z) = u$.

$$\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right] \qquad (1)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}(u^2) = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right] \qquad (2)$$

Adding (1) and (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right] + 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right\}\right] \qquad (3)$$

But u satisfies Laplace's equation and $f(z)$ is an analytic function, u and v satisfies C-R equations, that is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ and\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(3) becomes

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \qquad (4)$$

But $f'(z) = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ then $f'(\bar{z}) = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$

$$|f'(z)|^2 = f'(z)f'(\bar{z}) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

In view of this, the last gives

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$$

Result 5. If $u(x,y)$ and $v(x,y)$ are harmonic functions in a region R, prove that

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

is an analytic function of $z = x + iy$.

Proof: Suppose $u(x,y)$ and $v(x,y)$ satisfy Laplace's equation.

$$i.e., \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

$$and\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad (2)$$

Also suppose

$$s=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\ and\ t=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

To prove that $s+it$ is an analytic function, we have to show that

(i) $\frac{\partial s}{\partial x}=\frac{\partial t}{\partial y}\ and\ \frac{\partial s}{\partial y}=-\frac{\partial t}{\partial x}$

(ii) $\frac{\partial s}{\partial x},\frac{\partial t}{\partial y},\frac{\partial s}{\partial y}\ and\ -\frac{\partial t}{\partial x}$ are continuous

$$\frac{\partial s}{\partial x}-\frac{\partial t}{\partial y}=-\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0 \qquad (3)\ from\ (2)$$

$$\frac{\partial s}{\partial y}+\frac{\partial t}{\partial x}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0 \qquad (4)\ from\ (1)$$

From (3) and (4), the result (i) follows.

Existence of (1) and (2) implies that the result (ii).

Construction of an Analytic Function whose Real or Imaginary Part is known:

Let $f(z)=u+iv$ be an analytic function, whose real part u alone is known beforehand. We can find v, the imaginary part and also the function $f(z)$. The procedure is as follows:

First Method:

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

But $u\ and\ v$ are satisfy C-R equations

$$i.e.,\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ and\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Hence

$$dv=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy \qquad (1)$$

Now $\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)=-\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial x^2}$

As u satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$i.e.,\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}$$

Hence

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

and so the R.H.S. of (1) is a perfect differential.

Also $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ are known, since u is given.

Hence integrating (1), $v=\int\left(-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy\right)+c$

where c is an arbitrary constant. Thus v is known and the function $f(z)=u+iv$ is determined.

Second Method:

We know that $$f'(z)=\frac{\partial w}{\partial x}=\frac{\partial}{\partial x}(u+iv)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \qquad (1)$$

$$\left(\text{since } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \text{ from C} - \text{R equations}\right)$$

Since u is given, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ are known.

Hence integrating (1), $f(z)=\int\left(\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\right)dz+c.$

It is implied that the expression

$$\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$$

must be expressed in terms of $z=x+iy$, and then the above integration is to be effected.

Third Method (Milne-Thomson Method):

To find $f(z)$, when the real part $u(x,y)$ is given.

Let $f(z)=u(x,y)+i\,v(x,y) \qquad (1)$

Since $z=x+iy, \bar{z}=x-iy$, we have

$$x=\frac{z+\bar{z}}{2}, y=\frac{z-\bar{z}}{2i}$$

So $f(z)=u\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)+i\,v\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)$

Consider this as a formal identity in the two independent variables $z,\bar{z}$.

Putting $\bar{z}=z$, we get

$$f(z)=u(z,0)+i\,v(z,0) \qquad (2)$$

(1) Is the same as (1), if we replace $x\ by\ z\ and\ y\ by\ 0$.

$$\text{Now } f'(z) = \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(u+iv) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\left(\text{since } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ from C} - \text{R equations}\right)$$

$$\text{Let } \frac{\partial u}{\partial x} = \emptyset_1(x,y) \text{and } \frac{\partial u}{\partial y} = \emptyset_2(x,y).$$

Then $f'(z) = \emptyset_1(x,y) - i\,\emptyset_2(x,y)$ (3)

Now, to express $f'(z)$ completely in terms of z, we replace x by z and y by 0 in the expression (3).

$$f'(z) = \emptyset_1(z,0) - i\,\emptyset_2(z,0)$$

$$\text{Hence } f(z) = \int (\emptyset_1(z,0) - i\,\emptyset_2(z,0))dz \pm C$$

Similarly, given the imaginary part v, we can find u such that $u + iv$ is analytic. Let us use Milne-Thomson Method.

$$\text{Now } f'(z) = \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(u+iv) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$\left(\text{since } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ from C} - \text{R equations}\right)$$

$$f'(z) = \psi_1(x,y) - i\,\psi_2(x,y)$$

$$\text{where } \frac{\partial v}{\partial y} = \psi_1(x,y) \text{ and } \frac{\partial v}{\partial x} = \psi_2(x,y)$$

Now, to express $f'(z)$ completely in terms of z, we replace x by z and y by 0 in the above expression

$$f'(z) = \psi_1(z,0) - i\,\psi_2(z,0)$$

$$\text{Hence} \qquad f(z) = \int (\psi_1(z,0) - i\,\psi_2(z,0))dz \pm C.$$

EX. Show that the function $u = \frac{1}{2}log(x^2 + y^2)$ is harmonic and determine its conjugate.

Solution: Given $u = \frac{1}{2}log(x^2 + y^2)$

$$\text{We have } \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\text{Now } \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\text{Similarly,} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\text{Clearly,} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence u is harmonic. Let v be the conjugate of u. Then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \text{ (using } C - R \text{ equations)}$$

$$= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \frac{xdy - ydx}{x^2+y^2}$$

$$= \frac{x^2}{x^2+y^2}\left(\frac{xdy - ydx}{x^2}\right)$$

$$= \frac{x^2}{x^2+y^2}d\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}d\left(\frac{y}{x}\right) \quad = d\left(tan^{-1}\frac{y}{x}\right)$$

Hence Integrating

$$v = tan^{-1}\frac{y}{x} + C$$

EX. Find the analytic function whose real part is $\frac{x}{x^2+y^2}$.

Solution: Let $f(z) = u + iv$ where $u = \frac{x}{x^2+y^2}$

We have $\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$

Now

$$f^{'}(z) = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

$$= \frac{(y+ix)^2}{(x^2+y^2)^2} = \frac{(y+ix)^2}{(y+ix)^2(y-ix)^2}$$

$$= \frac{1}{(y-ix)^2} = \frac{i^2}{(iy+x)^2} = -\frac{1}{z^2}$$

Integrating, $f(z) = \frac{1}{z} + c$

EX. If $u - v = \frac{\cos\ x+sin\ x-e^{-y}}{2\ cos\ x-e^{y}-e^{-y}}$, when $f\left(\frac{\pi}{2}\right) = 0$ and $f(z)$ is analytic function of z, find $f(z)$ in terms of z.

Solution: Let $f(z) = u + iv$ (1)

So that $if(z) = iu - v$ (2)

Adding (1) and (2), we get

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$i.e., F(z) = U + iV \qquad (3)$$

Where $U = u - v\,, V = uv \; and \; F(z) = (1+i)f(z)$

If $f(z)$ is analytic, then $F(z)$ is also analytic.

$$\text{Given } U = u - v = \frac{\cos\, x + sin\, x - e^{-y}}{2\, cos\, x - e^{y} - e^{-y}} = \frac{\cos\, x + sin\, x - e^{-y}}{2\, (cos\, x - \; cosh\, y)}$$

We have

$$\frac{\partial U}{\partial x} = \frac{(\, cos\, x - cosh\, y).(\cos\, x - sin\, x) + (\cos\, x + sin\, x - e^{-y}).(sin\, x)}{2(cos\, x - cosh\, y)^2}$$

$$\frac{\partial U}{\partial y} = \frac{(cos\, x - cosh\, y).e^{-y} + (\cos\, x + sin\, x - e^{-y}).(sinh\, y)}{2(cos\, x - cosh\, y)^2}$$

$$\text{Now } F^{'}(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$= \frac{(\, cos\, x - cosh\, y).(\cos\, x - sin\, x) + (\cos\, x + sin\, x - e^{-y}).(sin\, x)}{2(cos\, x - cosh\, y)^2}$$

$$- i\frac{(cos\, x - cosh\, y).e^{-y} + (\cos\, x + sin\, x - e^{-y}).(sinh\, y)}{2(cos\, x - cosh\, y)^2}$$

By Milne-Thomson's method, $F^{'}(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence

$$F^{'}(z) = \frac{(cos\, z - 1).(cos\, z - sin\, z) + (cos\, z + sin\, z - 1).sin\, z}{2(cos\, z - 1)^2} - i\frac{(cos\, z - 1)}{2(cos\, z - 1)^2}$$

$$= (1+i)\frac{1}{2(1 - cos\, z)} = (1+i)\frac{1}{4}cosec^2\frac{z}{2}$$

Integrating,

$$F(z) = -\frac{(1+i)}{2}cot\,\frac{z}{2} + C$$

$$i.e., (1+i)f(z) = -\frac{(1+i)}{2}cot\,\frac{z}{2} + C$$

$$f(z) = -\frac{1}{2}cot\,\frac{z}{2} + c$$

Given $f\left(\frac{\pi}{2}\right) = 0$, then

$$f\left(\frac{\pi}{2}\right) = -\frac{1}{2} cot\ \frac{\pi}{4} + c$$

$$0 = -\frac{1}{2} + c$$

$$c = \frac{1}{2}$$

Hence

$$f(z) = \frac{1}{2}\left(1 - cot\ \frac{z}{2}\right)$$

Module 2

CONFORMAL MAPPING

Geometrical Representation of Functions of a Complex Variable:

We shall now consider the question of representing graphically the function of a complex variable. Real function of real variables, $y = f(x)$ can be exhibited graphically by plotting corresponding values of x and y as rectangular coordinates of points in the xy-plane. But in the complex domain, the functional relation

$$w = f(z) = f(x + iy) = u(x, y) + i\, v(x, y)$$

involves four real variables, namely, the two independent variables x and y and the two dependent variables u and v. Hence to plot $w = f(z)$ in the Cartesian fashion, a space of four dimensions is necessary. This is not possible instead, we make use of two complex Gauss planes, one for the variable $z = x + iy$ and the other for the variable $w = u + iv$. These are called the z-plane and the w-plane respectively. In the z-plane the point $x + iy$ is plotted and in the w-plane the point $u + iv$ is plotted. So in short, we have two Argand diagrams, one with the x and y variables and other with the u and v variables. A function $w = f(z)$ is now represented, not by a locus of points in a four dimensional region but by a correspondence between the points of these two planes. Corresponding to each point (x, y) in the z-plane for which $f(x + iy)$ is defined, there will be a point (u, v) in the w-plane, where $w = u + iv$.

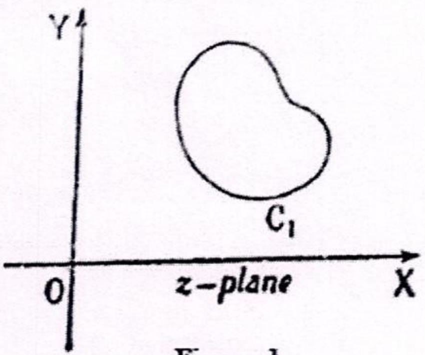


Figure 1

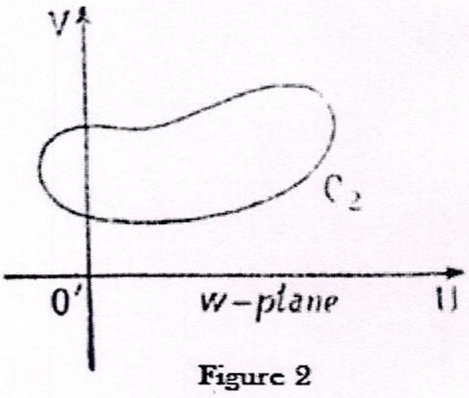


Figure 2

If the point z moves about in its plane along some curves, the point w will travel along a corresponding curve in its plane. If we describe a closed curve C_1 in the w-plane. We say that C_1 is mapped onto the corresponding curve C_2 in the w-plane by the function $w = f(z)$. The function $w = f(z)$ thus defines a mapping or a transformation if the figures of the z-plane into the figures on the w-plane. This function is called a ***transformation*** *or*

mapping function. The corresponding points, curves or regions in the two planes are called ***images*** of each other.

Even though two separate planes are used to represent w and z, it is often convenient to think of the mapping as effected in one plane, by using such technical terms as ***translation*** and ***rotation***. For example, the mapping by means of the function $w = z + 3$ moves every point z in the z-plane through 3 units to the right. However in the actual drawing of figures, the use of a separate w-plane serves to avoid confusion. To get a clear idea of the way in which the z-plane maps on to the w-plane, we choose various convenient families of curve in one plane and determine the following worked examples.

Conformal Mapping or Conformal Transformation:

The functional relationship$w = f(z)$ sets up a correspondence between the points $z = x + iy$ of the z-plane and $w = u + iv$ of the w-plane. Let two curves C_1 and C_2 of the z-plane be mapped as the curves $C_1^{'}$ and $C_2^{'}$ of the w-plane by the above transformation. $z = z_0$ the point of intersection of the curves C_1 and C_2 is mapped as $w = w_0$ the point of intersection of the curves $C_1^{'}$ and $C_2^{'}$. If the angle between the curves C_1 and C_2 at z_0 is the same, both in magnitude and sense as the angle between the curves $C_1^{'}$ and $C_2^{'}$ at w_0, the transformation is called ***conformal***.

Definition: A mapping or a transformation which preserves angles in magnitude and in sense between every pair of curves through a point is said to be *conformal* at that point.

Standard Transformations:

1. The transformation $w = z + c$ where c is a complex constant is a Translation.

Let the transformation $w = z + c$ where c is a complex constant nearly translates every point z through the constant vector representing c. Thus, if $z = x + iy$, $w = u + iv$, $c = c_1 + i\, c_2$, the equations of transformation are

$$u + iv = x + iy + c_1 + i\, c_2 = (x + c_1) + i(y + c_2)$$

$$i.e., u = x + c_1 \text{ and } v = y + c_2.$$

The image of any point (x, y) in the z-plane is the point $(u, v) i.e., (x + c_1 , y + c_2)$ in the w-plane. Every point in any region of the z-plane is mapped upon the w –plane in the same manner. It is clear that if the w –plane is superposed on the z-plane the figure shifted through a distance given by the vector c. Further the two regions have the same shape, size and orientation. In particular, this transformation changes circles into circles.

2. Expansion or contraction and rotation: (Magnification)

Consider the transformation $w = c\,z$, where c is a complex constant

Let $z = r\,e^{i\theta}, w = R\,e^{i\emptyset}$ and $c = \beta\,e^{i\alpha}$. Then

$$w = C\,z \Rightarrow R\,e^{i\emptyset} = \beta\,e^{i\alpha}.r\,e^{i\theta} = \beta r\,e^{i(\theta+\alpha)}$$

We have $R = \beta r$ and $\emptyset = \theta + \alpha$.

Thus under this transformation a point $P\,(r, \theta)$ in the z-plane is mapped to the point $P'(\beta r, \theta + \alpha)$ in the w-plane. Thus, this transformation effects an expansion when $|c| > 1$ and a contaction when $0 < |c| < 1$ of the radius vector by $\beta = |c|$ and rotation through an angle $\alpha = amp\ c$. Hence any figure in z-plane is transformed into, geometrically, a similar figure in the w-plane. In particular circles are mapped to circles.

Note 1. The above two transformations are the special cases of the transformation $w = az + c$, where $a\ and\ c$ are complex constants. Put $c = 0$, we get $w = az$ and by putting $a = 1$, we get $w = z + c$. This transformation $w = az + c$ is called a linear transformation.

Note 2. When $|c| = 1$ then $w = cz$ is called a pure rotation. Since in this case there is no expansion or contraction, but just a rotation through an angle of α.

Result: Circles are invariant under linear transformation.

Solution: Consider the linear transformation $w = az + c$, where $a\ and\ c$ are complex constants.

Consider the circle

$$A(x^2 + y^2) + Bx + Cy + D = 0 \qquad (1)$$

We have $w = u + iv = az + c = a(x + iy) + c_1 + i\,c_2$

$$= (ax + c_1) + i(ay + c_2)$$

Here $u = ax + c_1$ and $v = ay + c_2$. Then

$$x = \frac{u - c_1}{a}, y = \frac{v - c_2}{a} \qquad (2)$$

Substituting (2) in (1), we get

$$A'(u^2 + v^2) + B'u + C'v + D' = 0 \qquad (3)$$

Which is a circle in the w-plane, where

$$A' = \frac{A}{a^2}, B' = \frac{B - 2Ac_1}{a}, C' = \frac{C - 2Ac_2}{a}\ and\ D' = D + A\left(\frac{c_1^2 + c_2^2}{a^2}\right) - \frac{Bc_1}{a} - \frac{Cc_2}{a}$$

Hence the result.

3. The transformation $w = \frac{1}{z}$ (is called inversion)

Let $w = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$

$z\bar{z}$ is a real quantity. Hence length of $w = \frac{1}{|z|}$ and argument of $w =$ argument of $\bar{z}$.

To describe the geometrical process by which w is obtained, with the above characteristics from a given $\bar{z}$, we must first define the process of inversion.

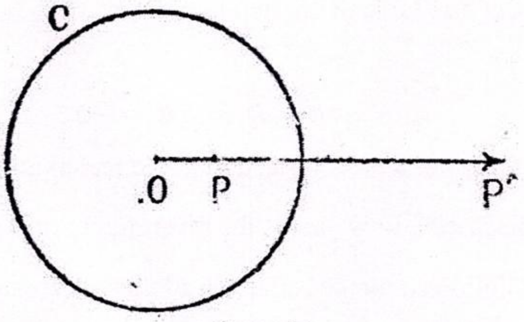


Figure 9

Let C be a circle with center O, radius r and P be any point in the plane of C. The inverse of P with respect to the circle C is defined to be the point P' on the radius OP such that $OP.OP' = r^2$. Obviously P is the inverse of P'. Now for the transformation $w = \frac{1}{z}$, first we must get a complex number whose length is $\frac{1}{|z|}$. If OP is the length corresponding to $|z|$ and OP' is the length corresponding to $\frac{1}{|z|}$, then

$$OP' = \frac{1}{OP} \; i.e., OP.OP' = 1$$

Hence P' is the inverse of P with respect to the unit circle. Secondly, if we take the reflection of P' in the real axis, we get a complex number with length $\frac{1}{|z|}$ and also having the direction of z.

Using polar coordinates, let

$$z = r\,e^{i\theta} \; and \; w = R\,e^{i\emptyset}$$

Then $R\,e^{i\emptyset} = \frac{1}{r\,e^{i\theta}} = \frac{e^{-i\theta}}{r}$

This may be split into successive transformation given by the equations $w_1 = \frac{e^{i\theta}}{r}$ and $w = w_1$

$$i.e., R_1.e^{i\emptyset_1} = \frac{e^{i\theta}}{r} \; and \; R\,e^{i\emptyset} = R_1.e^{-i\emptyset_1}$$

In the first transformation, we have

$$R_1 = \frac{1}{r} \; and \; \emptyset_1 = \theta$$

The argument of the image in the w_1-plane is the same as the argument of the given point in the z-plane and the product of the moduli =1. If the w_1-plane is superposed over the z-plane, the point P in the z-plane and its image P' in the w_1-plane will be collinear with the origin O such that $OP.OP' = 1$. So the first transformation $w_1 = \frac{e^{i\theta}}{r}$ gives the inverse of the given figure with respect to the unit circle. For the second transformation $w = w_1$, the equations are

$$R = R_1 \; and \; \emptyset = -\emptyset_1$$

i.e., the image of any point is its reflection in the real axis.

Hence the transformation $w = \frac{1}{z}$maps the given figure into firstly its inverse with respect to the unit circle followed by the reflection of the inverse into the real axis.

Discuss the transformation $w = z^2$.

Let $w = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$

Here$u = x^2 - y^2$ and$v = 2xy$

these are the equations of the transformation between the two planes. Using them, we can study various properties of the correspondence.

Case i: when $y =$ constant, a say, i.e., lines parallel to the x-axis mapping to curves in the w-plane whose parametric equations are

$u = x^2 - a^2$ and $v = 2ax$.

Eliminating the parameter x, we get

$$u = \frac{v^2}{4a^2} - a^2 \; i.e., v^2 = 4a^2(u + a^2)$$

This equation represents a family of parabolas having the origin of the w-plane as the focus, the u-axis, i.e., the line $v = 0$ as the axis and all extending to the right.

Case ii: when $x =$ constant, b say, i.e., lines parallel to the y-axis mapping to curves in the w-plane whose parametric equations are

$u = b^2 - y^2$ and $v = 2by$.

Eliminating the parameter y, we get

$$u = b^2 - \frac{v^2}{4b^2} \; i.e., v^2 = 4b^2(b^2 - u)$$

This equation represents a family of parabolas having the origin of the w-plane as the focus, $v = 0$ as the axis and all extending to the left. The plotting is illustrated in figures 12 and 13 for the values a, b equal to ½ and 1.

The rectangular region bounded by the lines $x = \frac{1}{2}, x = 1, y = \frac{1}{2}, y = 1$ maps on to the portion bounded by the four parabolas.

Also $w = z^2 \Rightarrow \frac{dw}{dz} = 2z = 0 \Rightarrow z = 0$

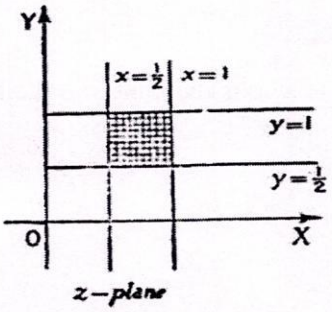
Therefore $z = 0$ is a critical point.

Using polar coordinates, let $z = r\, e^{i\theta}$ and $w = R\, e^{i\emptyset}$

Then $R\, e^{i\emptyset} = r^2 e^{2i\theta}$

Hence $R\ = r^2$ and $\emptyset = 2\theta$

The image of any point (r, θ) is that point in the w-plane whose polar coordinates are $R\ = r^2, \emptyset = 2\theta$.



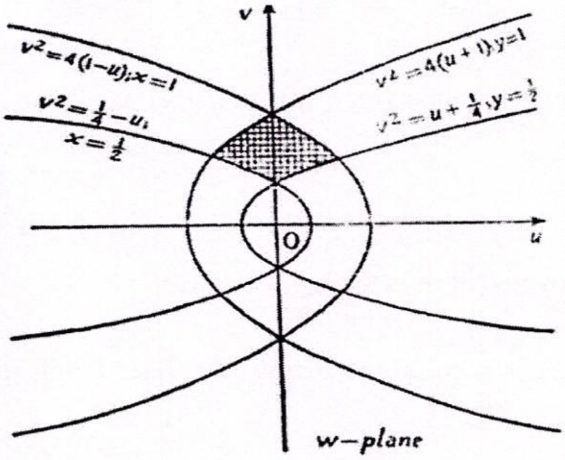


Figure 12 **Figure 13**

The upper half of the z-plane is defined by $0 < \theta < \pi$. Correspondingly, we have

$$0 < 2\theta < 2\pi \; i.e., 0 < \emptyset < 2\pi$$

which is the entire w-plane. So, the upper half of the z-plane transforms into the entire w-plane.

The first quadrant of the z-plane $(0 \leq \theta \leq \pi/2)$ maps into the entire upper half of the w-plane.

Circles about the origin $r = r_0$ are transformed into circles $R = r_0^2$ in the w-plane. A radial line $\theta = constant$ in the z-plane transforms into a new radial line $\emptyset = constant$ in the w-plane, with argument double that of the original line. The angle between two radial lines in the z-plane is doubled in the w-plane. The conformal property does not hold good at the origin. $z = 0$ is a critical point of the transformation.

The first quadrant of the semi circular region $r \leq r_0, 0 \leq \theta \leq \pi/2$is mapped into the upper half of the circular region $R \leq r_0^2$ as shown by broken lines, while the semi circular region $r \leq r_0$is mapped into the entire circular region $R \leq r_0^2$.

Discuss the transformation $w = e^z$.

Let $w = u + iv = e^z = e^{x+iy} = e^x(cos\, y + i\, sin\, y)$

Here $u = e^x cos\, y$ and $v = e^x\, sin\, y$

These are the equations of the transformation between the two planes. Using them, we can study various properties of the correspondence.

From the above $\frac{v}{u} = tan\, y$ (1)

and $u^2 + v^2 = e^{2x}$ (2)

Case i: when $y = constant, a\ say$, i.e., lines parallel to the x-axis mapping. So their images in the w-plane are

$$\frac{v}{u} = tan\, a = constant = k, say\ i.e., v = ku$$

These are a pencil of lines through the origin.

Case ii: when $x = constant, b\ say$, , i.e., lines parallel to the y-axis. So their images in the w-plane are

$u^2 + v^2 = e^{2b}$, which is a system of circles concentric with the origin.

In particular, the line $y = 0$ corresponds to $v = 0$.

The line $x = 0$ has its image the circle $u^2 + v^2 = 1$. The line $y = \frac{\pi}{4}$ corresponds to the line $\frac{v}{u} = tan\, \frac{\pi}{4}\ i.e., v = u$. The line $x = 1$ corresponds to the circle $u^2 + v^2 = e^2$.

Hence the rectangular region bounded by the lines $x = 0, x = 1, y = 0\ and\ y = \frac{\pi}{4}$ maps on to the sector of a ring as shown in figure bounded by the lines $v = 0, v = u$ and the circles

$$u^2 + v^2 = 1, u^2 + v^2 = e^2$$

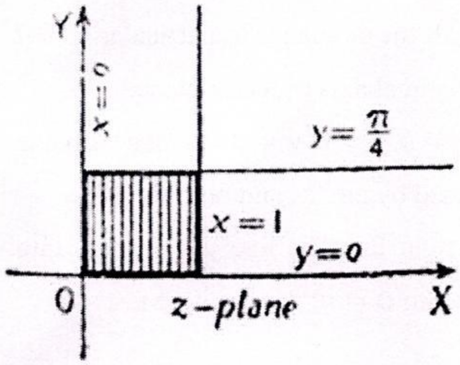


Figure 14

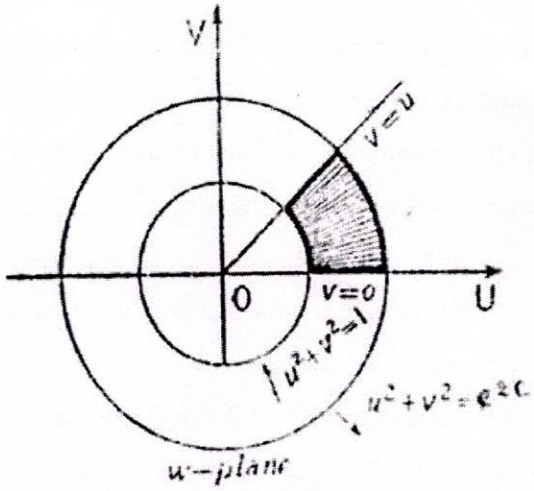


Figure 15

Taking $z = x + iy$ *and* $w = R\,e^{i\emptyset}$, we have

$$R\,e^{i\emptyset} = e^{x+iy} = e^{x}e^{iy}$$

$$\text{Hence } R = e^{x} \qquad (3)$$

$$\text{and } \emptyset = y \qquad (4)$$

Lines parallel to the y-axis in the z-plane, have the equation $x = constant = k$, say. Their images are given by (3) as $R = e^{k}$ which are concentric circles of the u-plane. If x is positive, $R > 1$ and if x is negative, $R < 1$. In particular, the imaginary axis $x = 0$ transforms to the unit circle in the w-plane.

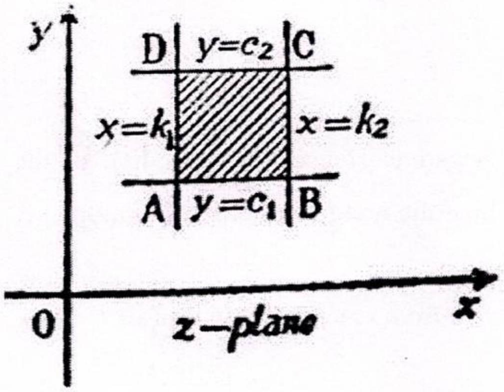


Figure 16

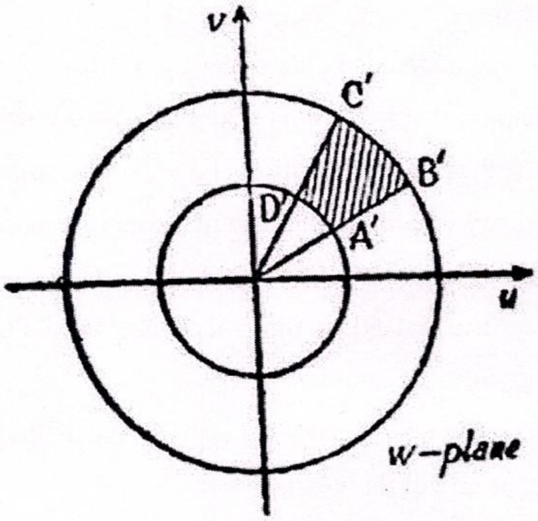


Figure 17

Horizontal lines in the z-plane have the equation $y = \text{constant} = c$, say. From (4) their images are given by $\emptyset = c$. These are radial lines in the w-plane. In particular $y = 0$ corresponds to $\emptyset = 0$. Hence, the x-axis transforms into the real axis of the w-plane.

The rectangular region bounded by $x = a, x = b, y = c, y = d$ in the z-plane transforms into the region $e^a \leq R \leq e^b, c \leq \emptyset \leq d$ bounded by circles and rays.

Consider the horizontal strip of the z-plane of height 2π. The line $y = 0$ maps into $\emptyset = 0$ i.e., the positive u-axis. The line $y = \pi$ transforms into $\emptyset = \pi$; the negative u-axis.

When $y = 2\pi, \emptyset = 2\pi$.

The radius vector $\emptyset = 2\pi$ in the w-plane is the same as the vector $\emptyset = 0$. Hence any horizontal strip of the z-plane of height 2π covers the entire w-plane once.

The Transformation $w = sin\ z$:

Let $w = u + iv = sin\ z = sin\ (x + iy)$

$$= sin\ x\ cos\ iy + cos\ x\ sin\ iy$$

$$= sin\ x\ cosh\ y + i\ cos\ x\ sinh\ y$$

$$\text{Hence } u = sin\ x\ cosh\ y \qquad (1)$$

$$\text{and } v = cos\ x\ sinh\ y \qquad (2)$$

We know that $sin\ z$ is periodic and hence it cannot be a one-to-one function, if considered in the entire z-plane.

Thus we restrict z to vertical infinite strip defined in $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Also $f'(z) = cos\ z, f'(z) = 0 \Rightarrow z = \pm\frac{\pi}{2}$ and these are two critical points for the conformal mapping.

The imaginary axis in the z-plane is $x = 0$.

When $x = 0, u = 0$ from (1) and $v = sinh\ y$ from (2).

If $y > 0, sinh\ y$ is positive and if $y < 0, sinh\ y$ is negative. Hence the upper half of the imaginary axis in the z-plane maps into the upper half of the imaginary axis of the w-plane, while the lower halves of both correspond with one another.

The real axis in the z-plane is $y = 0$. When $y = 0$ from (1) and (2), we get $u = sin\ x$ and $v = 0$.

Since $sin\ x$ takes values between -1 and 1, the image of the real axis $y = 0$ is the segment $-1 \leq u \leq 1$ of the u-axis.

Eliminating x from (1) and (2), we get

$$\frac{u^2}{cosh^2 y} + \frac{v^2}{sinh^2 y} = 1$$

when $y = constant = c$, say, we get

$$\frac{u^2}{cosh^2c} + \frac{v^2}{sinh^2c} = 1 \qquad (3)$$

when $c \neq 0$, (3) represents an ellipse with semi axes $cosh\ c\ and\ sinh\ c$. If c is the eccentricity of this ellipse, we have

$$e^2 = \frac{cosh^2c - sinh^2c}{cosh^2c} = \frac{1}{cosh^2c}$$

$$Hence\ e = \frac{1}{cosh\ c}$$

The foci are at

$$\left(\pm\ cosh\ c.\frac{1}{cosh\ c}, 0\right)\ i.e., at\ (1,0) and\ (-1,0).$$

These coordinates are independent of c and all the ellipses have the same foci.

Thus, lines parallel to the real axis of the z-plane map into confocal ellipses in the w-plane.

Since $cosh^2y - sinh^2y = 1$, eliminating y from (1) and (2), we get

$$\frac{u^2}{sin^2x} - \frac{v^2}{cos^2x} = 1$$

When $x = constant = k$, the above gives

$$\frac{u^2}{sin^2k} - \frac{v^2}{cos^2k} = 1 \qquad (4)$$

when $a \neq 0$(4)represents a system of hyper bolas having foci at $(1, 0)$ and $(-1, 0)$. Hence lines parallel the imaginary axes of the z-plane map into confocal hyperbolas.

The rectangle $ABCD$ in the z-plane, with sides along the lines $y = c, x = k, y = c', x = k'$ is transformed into the area $ABCD$ in the w-plane between the corresponding ellipses and hyperbolas. Actually there are four such areas but only one of these corresponds to the rectangle $ABCD$. The others are obtained from the areas which are the images of $ABCD$ in the x and y axes.

The Transformation $w = cos\ z$:

Let $w = u + iv = cos\ z = cos\ (x + iy)$

$$= cos\ x\ cos\ iy - sin\ x\ sin\ iy$$

$$= cos\ x\ cosh\ y - i\ sin\ x\ sinh\ y$$

$$Hence\ u = cos\ x\ cosh\ y \qquad (1)$$

$$and\ v = -sin\ x\ sinh\ y \qquad (2)$$

The imaginary axis in the z-plane is $x = 0$.

When $x = 0, u = cosh\ y$ from (1) and $v = 0$ from (2).

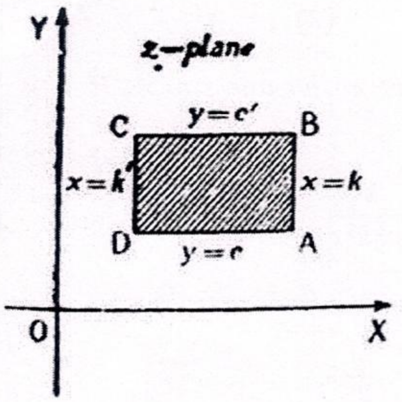


Figure 18

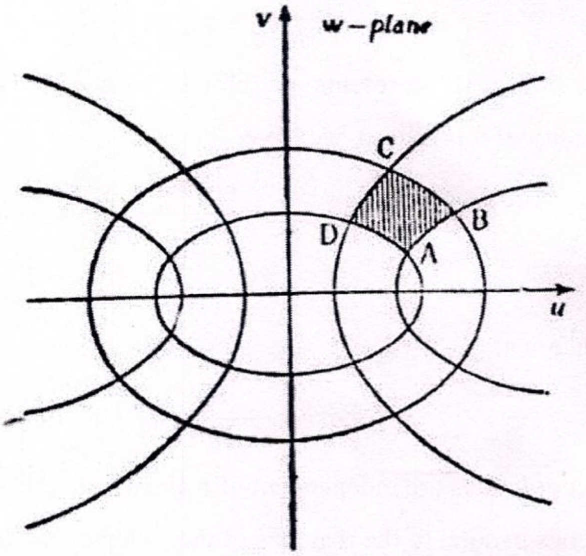


Figure 19

If $y > 0, sinh\, y$ is positive and if $y < 0, sinh\, y$ is negative. Hence the upper half of the imaginary axis in the z-plane maps into the lower half of the imaginary axis of the w-plane, while the lower half of the imaginary axis maps into the upper half of the imaginary axis.

The real axis in the z-plane is $y = 0$. When $y = 0$ from (1) and (2), we get $u = cos\, x$ and $v = 0$.

Since $cos\, x$ takes values between -1 and 1, the image of the real axis $y = 0$ is the segment $-1 \leq u \leq 1$ of the u-axis.

Eliminating x from (1) and (2), we get

$$\frac{u^2}{cosh^2 y} + \frac{v^2}{sinh^2 y} = 1$$

When $y = constant = c$, say, we get

$$\frac{u^2}{cosh^2 c} + \frac{v^2}{sinh^2 c} = 1 \qquad (3)$$

When $c \neq 0$, (3) represents an ellipse with semi axes $cosh\, c\ and\ sinh\, c$. If c is the eccentricity of this ellipse, we have

$$e^2 = \frac{cosh^2 c - sinh^2 c}{cosh^2 c} = \frac{1}{cosh^2 c}$$

$$Hence\ e = \frac{1}{cosh\ c}$$

The foci are at